SUPPLEMENT TO
"SYMMETRIC PROPOSITIONS AND LOGICAL QUANTIFIERS"

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Abstract. This document contains propositions, proofs, and supplementary discussion supporting claims made in the paper "Symmetric propositions and logical quantifiers," ***Journal of philosophical logic***, forthcoming (henceforth SPLQ). Unless explicitly mentioned otherwise, all references are internal. Section, but not subsection, headings correspond to those of SPLQ. (The latter has no subsections.)

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§1. Introduction.

§2. Zermelo systems and Zermelo logics. Relative to a collection of urelements, we assume, as background context, both a model V^* of ZF^2 with urelements as well as, within it, an unbounded sequence $\{V^*_{\theta_\kappa}\}_{\kappa\in On}$ of natural models of ZF^2; and we give precedence to these natural models. Operationally, this means that any collection somehow bounded by inaccessible θ is a set within a natural model $\langle U, V^*_\theta, \in|V^*_\theta\rangle$ having appropriate urelements. On the other hand, if a collection is indexed by such a θ, then we must look to $\langle U, V^*_{\theta'}, \in|V^*_{\theta'}\rangle$ with inaccessible $\theta' > \theta$. In particular, any Zermelo logic is a set in some natural model of ZF^2 having fundamental propositions as urelements essentially, whereas entire Zermelo systems will always make for proper classes even within $V^* =: \bigcup_{\alpha\in\mathrm{On}} V^*_\alpha$. Thus any logic $\mathcal{H}^\theta_{\mathfrak{D},\Sigma}$ can be situated within some model of a well-understood theory and is, to this extent, legitimated: $\mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ is not "too big" and will not give rise to contradiction assuming that ZF^2 itself is consistent.

Lemma 2.1. *Let θ be inaccessible and suppose that $\mathcal{A} \in V_\theta$. Then* $\mathrm{rank}(\mathcal{A})$ *is not a limit ordinal.*

Proof. Let $\alpha \leq \theta$ be the least limit ordinal with $\mathcal{A} \in V_\alpha$. By the definition of $\{V_\alpha\}_{\alpha\in\mathrm{On}}$ in §2 of SPLQ, we have that $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ so that $\mathrm{rank}(\mathcal{A}) < \alpha$ cannot be a limit. $\dashv$

Theorem 2.2. *Let φ be the unary function defined in §2 of SPLQ.*

1. *For all ordinals ζ, we have $\zeta \leq \varphi(\zeta)$.*
2. *If $\xi < \zeta$, then $\varphi(\xi) < \varphi(\zeta)$. That is, φ is strictly increasing.*
3. *For all ordinals ζ, we have $2^\zeta \leq \varphi(\zeta+1)$.*
4. *Let θ be inaccessible. Then $\xi < \theta$ implies $\varphi(\xi) < \theta$.*
5. *If θ is inaccessible, then $\theta = \varphi(\theta)$.*

Proof. The proofs of (1) and (4) proceed by transfinite induction.

1. First, $0 \leq 1 = \varphi(0)$. If $\zeta = \xi + 1$, then
$$\zeta = \xi + 1 \leq \varphi(\xi) + 1 \leq 2^{\varphi(\xi)} = \varphi(\zeta).$$
Otherwise, $\zeta = \bigcup_{\xi<\zeta} \xi \leq \bigcup_{\xi<\zeta} \varphi(\xi) = \varphi(\zeta)$.
2. It is enough to see that, for arbitrary ξ, we have $\varphi(\xi) < 2^{\varphi(\xi)} = \varphi(\xi+1)$.
3. By (1) we have $2^\zeta \leq 2^{\varphi(\zeta)}$, and the right member here is $\varphi(\zeta+1)$.
4. First, $\varphi(0) = 1 < \theta$. Next, if $\xi = \xi'+1 < \theta$, then $\varphi(\xi) = 2^{\varphi(\xi')} < \theta$ since, by θ a strong limit, $\varphi(\xi') < \theta$ implies $2^{\varphi(\xi')} < \theta$. Finally, if $\xi = \bigcup_{\xi'<\xi} \xi' < \theta$, then $\varphi(\xi) = \bigcup_{\xi'<\xi} \varphi(\xi') < \theta$ by θ regular.
5. Suppose θ is inaccessible. Then $\theta = \bigcup_{\xi<\theta} \xi$. So $\varphi(\theta) = \varphi(\bigcup_{\xi<\theta} \xi) = \bigcup_{\xi<\theta} \varphi(\xi) \leq \theta$ by (4). So $\theta = \varphi(\theta)$ by (1).

$\dashv$

Lemma 2.3. *Let θ be inaccessible and let Γ be a class of ordinals each strictly less than θ with $|\Gamma| < \theta$. Then* $\sup \Gamma < \theta$.

Proof. First, since $|\Gamma|$ is a set, by Replacement so is Γ. But then there exists $f : |\Gamma| \mapsto \Gamma$ increasing. If $\sup \Gamma = \theta$, we would have $\mathrm{cf}\ \theta \leq |\Gamma|$, contradicting θ regular. $\dashv$

What follows is Theorem 2.1 of SPLQ. (Recall that $\mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ is V_θ.)

Theorem 2.4. *Let domain $\mathfrak{D}$ and signature Σ be given. Suppose that θ is inaccessible. Then for arbitrary $K \subseteq V_\theta$ we have $\bigvee K, \bigwedge K \in V_\theta$ if and only if $|K| < \theta$.*

Proof. Suppose $K \subseteq V_\theta$, where $|K| < \theta$. Let $\Gamma = \{\gamma \mid$ there exists $\mathcal{A} \in K$ with $\mathcal{A}$ of rank $\gamma\}$ and set $\rho = \sup \Gamma$. It follows that $K \subseteq V_\rho$. Since $|K| < \theta$ and since, by Lemma 2.1, no element of Γ is a limit, we have $\rho < \theta$ by Lemma 2.3. Further, $\varphi(|K|) < \theta$ by Theorem 2.2.6.

- If $\rho \leq |K|$, then we have $K \subseteq V_\rho \subseteq V_{|K|}$ and $|K| \leq \varphi(|K|)$. It follows that $\bigvee K \in V_{|K|+1}$.

- If $|K| < \rho$, then $K \subseteq V_\rho \subseteq V_{\rho+1}$ and $|K| < 2^\rho \leq \varphi(\rho+1)$ by Theorem 2.2.5. So $\bigvee K \in V_{\rho+2}$.

We conclude that $\operatorname{rank}(\bigvee K) \leq \max(|K|, \rho+1) + 1 < \theta$.

For the other direction, if $|K| \geq \theta$, then by Theorems 2.2.2 and 2.2.5 we have that $|K| \geq \varphi(\theta)$ as well. Suppose, for the sake of producing a contradiction, that $\bigvee K \in V_\theta$. By Lemma 2.1 the rank of $\bigvee K$ is some successor $\beta + 1 < \theta$. But then by the definition of $V_{\beta+1}$ and Theorem 2.2.4, we have $|K| \leq \varphi(\beta) < \theta$.

Finally, suppose $K \subseteq V_\theta$ with $\bigwedge K =: \neg(\bigvee\{\neg(A) \mid \mathcal{A} \in K\}) \in V_\theta$. It follows that $\bigvee\{\neg(A) \mid \mathcal{A} \in K\} \in V_\theta$ also. By the foregoing, $|K| = |\{\neg(A) \mid \mathcal{A} \in K\}| < \theta$. The other direction is similar. ⊣

2.1. The equivalence of two characterizations of $\mathcal{H}_{\mathfrak{D},\Sigma}$. First, where φ is the unary cardinal-valued function defined by

$$\varphi(\zeta) =: \begin{cases} 0 & \text{if } \zeta = 0 \\ 2^{\varphi(\xi)} & \text{if } \zeta = \xi + 1 \\ \bigcup_{\xi<\zeta} \varphi(\xi) & \text{otherwise,} \end{cases}$$

we define $\mathcal{H}_{\mathfrak{D},\Sigma} =: \bigcup_{\alpha \in \mathrm{On}} V_\alpha$ where

$$(\star) \qquad V_\alpha =: \begin{cases} \mathfrak{G}_{\mathfrak{D},\Sigma} & \text{if } \alpha = 0 \\ \{\neg\mathcal{A} \mid \mathcal{A} \in V_\beta\} \cup \{\bigvee K \mid K \subseteq V_\beta \wedge |K| \leq \varphi(\alpha)\} & \text{if } \alpha = \beta + 1 \\ \bigcup_{\beta<\alpha} V_\beta & \text{otherwise.} \end{cases}$$

Next, Zermelo's own hierarchy $\mathcal{H}'_{\mathfrak{D},\Sigma} =: \bigcup_{0<\alpha} V'_\alpha$ as in [15] where

$$(+) \qquad V'_\alpha =: \begin{cases} \mathfrak{G}_{\mathfrak{D},\Sigma} & \text{if } \alpha = 1 \\ V'_\beta \cup \{\neg\mathcal{A} \mid \mathcal{A} \in V'_\beta\} \cup & \\ \qquad \{\bigvee K \mid K \subseteq V'_\beta\} \cup \{\bigwedge K \mid K \subseteq V'_\beta\} & \text{if } \alpha = \beta + 1 > 1 \\ \bigcup_{0<\beta<\alpha} V'_\beta & \text{otherwise.} \end{cases}$$

Regarding our use of $(\star)$ rather than $(+)$, we mention that only $(+)$ supports Zermelo's talk of noncumulative levels $Q_\alpha =: V'_{\alpha+1} \setminus V'_\alpha$ for $\alpha > 0$ as "levels of quantification" (*Stufen der Quantifikation*). (We let Q_0 be $\mathfrak{G}_{\mathfrak{D},\Sigma}$.)

It is easily shown that $(\star)$ and $(+)$ describe equivalent systems of propositions in the following sense.

Theorem 2.5. *Let domain $\mathfrak{D}$ and signature Σ be given. Then, for any strongly inaccessible θ such that $|\mathfrak{D}| < \theta$, we have that $V_\theta = V'_\theta$.*

Proof. By induction on the complexity (or rank) of $\mathcal{A} \in V'_\theta$, we show that $\mathcal{A} \in V_\theta$, thereby establishing $V'_\theta \subseteq V_\theta$. We mention at the outset that $|\mathfrak{D}| < \theta$ implies $|\mathfrak{G}_{\mathfrak{D},\Sigma}| < \theta$, which in turn implies $|V'_\beta| < \theta$ for all $0 < \beta < \theta$.

1. Suppose that $\mathcal{A} \in V'_\theta$ is elementary $R_\ell \mathfrak{a}_1 \ldots \mathfrak{a}_{n_\ell}$. Then $\mathcal{A} \in V'_1 = \mathfrak{G}_{\mathfrak{D},\Sigma} = V_0 \subseteq V_\theta$.
2. Suppose that $\mathcal{A} \in V'_\theta$ is of the form $\neg(\mathcal{B}) \in V'_{\beta+1}$ with $\mathcal{B} \in V'_\beta$ for some $0 < \beta < \theta$. By induction hypothesis we have $\mathcal{B} \in V_\theta$ and hence in V_γ for some $\gamma < \theta$. But then $\mathcal{A} \in V_{\gamma+1} \subseteq V_\theta$.

3. Suppose that $\mathcal{A} \in V'_\theta$ is of the form $\bigvee K \in V'_{\beta+1}$ with $K \subseteq V'_\beta$ for some $0 < \beta < \theta$. Since $|V'_\beta| < \theta$, we have $|K| < \theta$. By induction hypothesis we have $K \subseteq V_\theta$, and, since $|K| < \theta$, we have $\mathcal{A} \in V_\theta$ by Theorem 2.4.
4. Suppose that $\bigwedge K \in V'_\theta$ so that $K \subseteq V'_\theta$ and let $K^\neg =: \{\neg(\mathcal{A}) \mid \mathcal{A} \in K\}$. We have $|K^\neg| = |K| < \theta$ as in (3) and $K^\neg \subseteq V_\theta$ also so that $\bigvee K^\neg \in V_\theta$ by Theorem 2.4. Then $\neg \bigvee K^\neg \in V_\theta$.

For the other direction, suppose that $\mathcal{A} \in V_\theta$. If $\mathcal{A}$ is either atomic or of the form $\neg(\mathcal{B})$, then reasoning as in (1)–(2) above suffices to see that $\mathcal{A} \in V'_\theta$. Suppose that $\mathcal{A} \in V_\theta$ is of the form $\bigvee K$. By Theorem 2.4 we have $K \subseteq V_\theta$ with $|K| < \theta$. Also $K \subseteq V'_\theta$ by induction hypothesis, and $|K| < \theta$ implies $K \subseteq V'_\beta$ for some $0 < \beta < \theta$. So $\bigvee K \in V'_{\beta+1} \subseteq V'_\theta$ by (+), and we are done. ⊣

§3. A semantics based on Boolean-valued functions.

Theorem 3.1. *Let domain $\mathfrak{D}$ and signature Σ be given.*
1. *If $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},\Sigma}$, then* $\mathrm{Mod}(\neg\mathcal{A}) = \overline{\mathrm{Mod}(\mathcal{A})} =: \mathfrak{T}^{\vec{n}}_{\mathfrak{D}} \setminus \mathrm{Mod}(\mathcal{A})$.
2. *If $K \subseteq V_\beta$ and $|K| \leq \varphi(\beta+1)$, then we have* $\mathrm{Mod}(\bigvee K) = \bigcup_{\mathcal{B}\in K} \mathrm{Mod}(\mathcal{B})$.

Proof. We take each part in turn.
1. By Definition 3.2 of SPLQ we have the following.

$$\begin{aligned}\tau \in \mathrm{Mod}(\neg\mathcal{A}) &\iff \tau \models \neg\mathcal{A}\\ &\iff \mathrm{val}(\neg\mathcal{A},\tau) = \mathrm{true}\\ &\iff \mathrm{val}(\mathcal{A},\tau) = \mathrm{false}\\ &\iff \tau \not\models \mathcal{A}\\ &\iff \tau \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}} \setminus \mathrm{Mod}(\mathcal{A}).\end{aligned}$$

2. By hypothesis and Theorems 2.2.4 and 2.4 we have $\bigvee K \in \mathcal{H}_{\mathfrak{D},\Sigma}$. Further, by Definition 3.2 of SPLQ again, we have

$$\begin{aligned}\tau \in \mathrm{Mod}(\bigvee K) &\iff \tau \models \bigvee K\\ &\iff \mathrm{val}(\bigvee K,\tau) = \mathrm{true}\\ &\iff \mathrm{val}(\mathcal{B},\tau) = \mathrm{true} \text{ for some } \mathcal{B} \in K\\ &\iff \tau \models \mathcal{B} \text{ for some } \mathcal{B} \in K\\ &\iff \tau \in \mathrm{Mod}(\mathcal{B}) \text{ for some } \mathcal{B} \in K\\ &\iff \tau \in \bigcup_{\mathcal{B}\in K} \mathrm{Mod}(\mathcal{B}).\end{aligned}$$

⊣

That $\mathcal{A} \Leftrightarrow \bigvee\{\mathrm{eldg}(\tau) \mid \tau \in \mathrm{Mod}(\mathcal{A})\}$ is mentioned in the fifth paragraph of [14].

Theorem 3.2 (Theorem 3.3 of SPLQ). *Suppose that domain $\mathfrak{D}$ and signature $\Sigma = \langle R_1, \ldots, R_p\rangle$ are given. Then $\tau \in \mathrm{Mod}(\mathcal{A})$ if and only if $\mathrm{eldg}(\tau) \Rightarrow \mathcal{A}$.*

Proof. Suppose $\tau \in \mathrm{Mod}(\mathcal{A})$. Then $\mathrm{eldg}(\tau) \Rightarrow \bigvee_{\tau\in\mathrm{Mod}(\mathcal{A})} \mathrm{eldg}(\tau)$ holds, and we have $\bigvee_{\tau\in\mathrm{Mod}(\mathcal{A})} \mathrm{eldg}(\tau) \Rightarrow \mathcal{A}$ by the remark preceding the statement of

Theorem 3.3 of SPLQ. As for the other direction, $\mathrm{Mod}(\mathrm{eldg}(\tau)) = \{\tau\}$ and by assumption $\mathrm{Mod}(\mathrm{eldg}(\tau)) \subseteq \mathrm{Mod}(\mathcal{A})$. $\dashv$

Regarding the quotient algebra introduced at the very end of §3 of SPLQ, we have the following definitions. Suppose that $\mathfrak{K}$ is the set of equivalence classes of members of $K \subseteq \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ satisfying $|K| < \theta$. Then we write $\bigvee \mathfrak{K}$ for $[\bigvee K]_{\equiv}$ and $\bigwedge \mathfrak{K}$ for $[\bigwedge K]_{\equiv}$. Otherwise, $\neg[\mathcal{A}]_{\equiv}$ is $[\neg\mathcal{A}]_{\equiv}$, and $\perp$ and $\top$ are $[\perp]_{\equiv}$ and $[\top]_{\equiv}$, respectively. We write $[\mathcal{A}]_{\equiv} \vee [\mathcal{B}]_{\equiv}$ for $\bigvee\{[\mathcal{A}]_{\equiv}, [\mathcal{B}]_{\equiv}\}$ and $[\mathcal{A}]_{\equiv} \wedge [\mathcal{B}]_{\equiv}$ for $\bigwedge\{[\mathcal{A}]_{\equiv}, [\mathcal{B}]_{\equiv}\}$.

The mapping $\varphi : \mathcal{H}_{\mathfrak{D},\Sigma}/{\equiv} \mapsto \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}/{\equiv}$ defined by $\varphi([\mathcal{A}]_{\equiv}) = [\mathcal{A}]_{\equiv} \cap \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ is an isomorphism of the smallest θ-complete subalgebra of $\langle \mathcal{H}_{\mathfrak{D},\Sigma}/{\equiv}, \bigvee, \bigwedge, \neg, \perp, \top\rangle$ containing each elementary proposition onto $\langle \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}/{\equiv}, \bigvee, \bigwedge, \neg, \perp, \top\rangle$. Suppose that $|\mathfrak{D}| < \theta$. Then it is obvious that the atoms of $\langle \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}/{\equiv}, \bigvee, \bigwedge, \neg, \perp, \top\rangle$ are precisely the unitary propositions over $\mathfrak{D}$ and Σ, all of which are members of $\mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$. By Theorem 3.2, for each satisfiable $\mathcal{A} \in \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$, there exists an atom $\mathcal{B}$ with $\mathcal{B} \Rightarrow \mathcal{A}$, which means that $\langle \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}/{\equiv}, \bigvee, \bigwedge, \neg, \perp, \top\rangle$ is atomic (see [4], p. 147). If, instead, $|\mathfrak{D}| \geq \theta$, then $\mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}/{\equiv}$ is seen to contain no atoms, where we can reason as follows.

First, by an easy inductive argument we can show that the constituent set $\mathrm{Constit}(\mathcal{A})$ of any $\mathcal{A} \in \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ satisfies $|\mathrm{Constit}(\mathcal{A})| < \theta$ (cf. Definition 4.10). Suppose, for the sake of contradiction, that $\mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}/{\equiv}$ contains an atom, that is, an element $[\mathcal{A}]_{\equiv} \neq [\perp]_{\equiv}$ such that $[\mathcal{B}]_{\equiv} \preceq [\mathcal{A}]_{\equiv}$ implies either $\mathcal{B}$ unsatisfiable or $\mathcal{A} \Leftrightarrow \mathcal{B}$. Since $|\mathfrak{D}| \geq \theta$ by assumption, there exists $\mathfrak{a} \in \mathfrak{D} \setminus \mathrm{Constit}(\mathcal{A}) \neq \emptyset$. By Theorem 4.13 it can be seen that $\mathcal{A} \wedge R_\ell \mathfrak{a} \ldots \mathfrak{a}$ is satisfiable with $\mathcal{A} \wedge R_\ell \mathfrak{a} \ldots \mathfrak{a} \not\Leftrightarrow \mathcal{A}$, where $1 \leq \ell \leq p$ and $R_\ell \mathfrak{a} \ldots \mathfrak{a} \in \mathfrak{G}_{\mathfrak{D},\Sigma} \subseteq \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$. It follows that $[\perp]_{\equiv} \prec [\mathcal{A} \wedge R_\ell \mathfrak{a} \ldots \mathfrak{a}]_{\equiv} \prec [\mathcal{A}]_{\equiv}$, which contradicts $[\mathcal{A}]_{\equiv}$ an atom.

§4. Symmetric and categorical propositions. Immediately after Definition 4.1 of SPLQ, it is mentioned that any proposition over a single-element domain is symmetric, which brings to mind the fact that any polynomial function in a single variable is a symmetric function.

If $\mathfrak{C}$ is an invariant subset of $\mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$ and $\{\mathfrak{C}_\beta\}_{\beta<\gamma}$ is a family of such subsets, then $\overline{\mathfrak{C}}$, $\bigcup_{\beta<\gamma} \mathfrak{C}_\beta$, and $\bigcap_{\beta<\gamma} \mathfrak{C}_\beta$ are all invariant. Consequently, we have

THEOREM 4.1. *Let domain $\mathfrak{D}$ and signature Σ be given. If $\mathcal{A}$ and each member of family $\{\mathcal{A}_\beta\}_{\beta<\gamma}$ are in $\mathrm{Sym}_{\mathfrak{D},\Sigma}$, then so are $\neg\mathcal{A}$, $\bigvee_{\beta<\gamma} \mathcal{A}_\beta$, and $\bigwedge_{\beta<\gamma} \mathcal{A}_\beta$.*

Since any $K \subseteq \mathcal{H}_{\mathfrak{D},\Sigma}$ constitutes a well-ordered family of propositions, Theorem 4.1 entails that, with $K \subseteq \mathrm{Sym}_{\mathfrak{D},\Sigma}$, both $\bigvee K, \bigwedge K \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$ (cf. Theorem 4.2 of SPLQ).

Theorem 4.2 of SPLQ refers in effect to system $\mathcal{H}_{\mathfrak{D},\Sigma}$. The analogue for logic $\mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ follows from the definition of $\mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma}$ together with Theorem 2.4 and Theorem 4.2 of SPLQ: if $\mathcal{A}$ and each member of K with $|K| < \theta$ are in $\mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma}$, then so are $\neg\mathcal{A}$, $\bigvee K$, and $\bigwedge K$. For simplicity we usually present only system versions of definitions and theorems and regularly appeal to the system version even when the corresponding logic version is more to the point.

THEOREM 4.2. *Where* $\mathfrak{D}$ *is any singleton domain, proposition* $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},\Sigma}$ *is categorical over* $\mathfrak{D}$ *and arbitrary* Σ *just in case* $|\mathrm{Mod}(\mathcal{A})| = 1$.

PROOF. If $|\mathrm{Mod}(\mathcal{A})| = 1$, then $\mathrm{Mod}(\mathcal{A})$ has no nonempty proper subsets, and we are done. For the other direction, by assumption, there are no nontrivial domain permutations. Consequently, if $|\mathrm{Mod}(\mathcal{A})| > 1$, then any nonempty proper subset is permutation-invariant. And if $|\mathrm{Mod}(\mathcal{A})| < 1$, then $\mathcal{A}$ is contradictory and hence noncategorical. ⊣

We justify the remark following Definition 4.4 of SPLQ.

THEOREM 4.3. *Let* $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},\Sigma}$. *Then* $\mathcal{A} \in \mathrm{Cat}_{\mathfrak{D},\Sigma}$ *if and only if* $\mathcal{A} \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$ *is satisfiable and any satisfiable* $\mathcal{B} \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$ *with* $\mathcal{B} \Rightarrow \mathcal{A}$ *satisfies* $\mathcal{A} \Leftrightarrow \mathcal{B}$ *in fact.*

PROOF. Let $\mathcal{A} \in \mathrm{Cat}_{\mathfrak{D},\Sigma}$. Since its model set is then nonempty and invariant by definition, it follows immediately that $\mathcal{A}$ is satisfiable and symmetric. Suppose, for the sake of contradiction, that there exists satisfiable symmetric $\mathcal{B}$ with $\mathcal{B} \Rightarrow \mathcal{A}$ but $\mathcal{B} \not\Leftrightarrow \mathcal{A}$. Then $\emptyset \neq \mathrm{Mod}(\mathcal{B}) \subsetneq \mathrm{Mod}(\mathcal{A})$ so that minimal invariant $\mathrm{Mod}(\mathcal{A})$ has a nonempty invariant proper subset, which is absurd.

For the other direction, suppose that $\mathcal{A}$ is satisfiable and symmetric but not categorical. It follows that $\mathrm{Mod}(\mathcal{A})$ has a nonempty invariant proper subset $\mathfrak{C} \subseteq \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$. Since $\mathrm{Mod}\big(\bigvee_{\tau\in\mathfrak{C}} \mathrm{eldg}(\tau)\big) = \mathfrak{C} \neq \emptyset$ is invariant, we have $\bigvee_{\tau\in\mathfrak{C}} \mathrm{eldg}(\tau)$ satisfiable and symmetric. Since $\mathrm{Mod}\big(\bigvee_{\tau\in\mathfrak{C}} \mathrm{eldg}(\tau)\big) = \mathfrak{C} \subsetneq \mathrm{Mod}(\mathcal{A})$, we have that $\bigvee_{\tau\in\mathfrak{C}} \mathrm{eldg}(\tau) \Rightarrow \mathcal{A}$ but $\bigvee_{\tau\in\mathfrak{C}} \mathrm{eldg}(\tau) \not\Leftrightarrow \mathcal{A}$. ⊣

The logic version of Theorem 4.3 splits into two cases, depending upon the cardinality of $\mathfrak{D}$. Both logic versions are consequences of Theorem 4.3.

THEOREM 4.4. *Let* $\mathcal{A} \in \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ *with* $|\mathfrak{D}| < \theta$. *Then* $\mathcal{A} \in \mathrm{Cat}^{\theta}_{\mathfrak{D},\Sigma}$ *if and only if* $\mathcal{A} \in \mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma}$ *is satisfiable and any satisfiable* $\mathcal{B} \in \mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma}$ *with* $\mathcal{B} \Rightarrow \mathcal{A}$ *satisfies* $\mathcal{A} \Leftrightarrow \mathcal{B}$ *in fact.*

PROOF. Let $\mathcal{A} \in \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$. Suppose $\mathcal{A} \in \mathrm{Cat}^{\theta}_{\mathfrak{D},\Sigma}$. Then $\mathcal{A} \in \mathrm{Cat}_{\mathfrak{D},\Sigma}$ and by Theorem 4.3 we have that $\mathcal{A} \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$ is satisfiable. So $\mathcal{A} \in \mathrm{Sym}_{\mathfrak{D},\Sigma} \cap \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma} = \mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma}$ as well. Suppose that $\mathcal{B} \in \mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma}$ satisfiable is such that $\mathcal{B} \Rightarrow \mathcal{A}$. Then $\mathcal{B} \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$ also so that $\mathcal{A} \Leftrightarrow \mathcal{B}$.

For the other direction, note first that $|\mathfrak{D}| < \theta$ implies $|\mathfrak{T}^{\vec{n}}_{\mathfrak{D}}| < \theta$. Suppose that $\mathcal{A} \in \mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma} \setminus \mathrm{Cat}^{\theta}_{\mathfrak{D},\Sigma}$ is satisfiable. Then $\mathcal{A} \notin \mathrm{Cat}_{\mathfrak{D},\Sigma}$ since $\mathcal{A} \in \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ by assumption. (Recall that $\mathrm{Cat}^{\theta}_{\mathfrak{D},\Sigma} =: \mathrm{Cat}_{\mathfrak{D},\Sigma} \cap \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$.) So $\mathrm{Mod}(\mathcal{A})$ has a nonempty invariant proper subset $\mathfrak{C} \subseteq \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$. Since $\mathrm{Mod}\big(\bigvee_{\tau\in\mathfrak{C}} \mathrm{eldg}(\tau)\big) = \mathfrak{C} \neq \emptyset$ is invariant, we have $\bigvee_{\tau\in\mathfrak{C}} \mathrm{eldg}(\tau) \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$ satisfiable. Since $\mathrm{Mod}\big(\bigvee_{\tau\in\mathfrak{C}} \mathrm{eldg}(\tau)\big) = \mathfrak{C} \subsetneq \mathrm{Mod}(\mathcal{A})$, we have $\bigvee_{\tau\in\mathfrak{C}} \mathrm{eldg}(\tau) \Rightarrow \mathcal{A}$ but $\bigvee_{\tau\in\mathfrak{C}} \mathrm{eldg}(\tau) \not\Leftrightarrow \mathcal{A}$. Moreover, $|\mathfrak{C}| < \theta$ gives $\bigvee_{\tau\in\mathfrak{C}} \mathrm{eldg}(\tau) \in \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$. So $\bigvee_{\tau\in\mathfrak{C}} \mathrm{eldg}(\tau) \in \mathrm{Sym}_{\mathfrak{D},\Sigma} \cap \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma} = \mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma}$. ⊣

THEOREM 4.5. *Let* $|\mathfrak{D}| \geq \theta$. *Then* $\mathrm{Cat}^{\theta}_{\mathfrak{D},\Sigma} = \emptyset$.

PROOF. Suppose for the sake of contradiction that $\mathcal{A} \in \mathrm{Cat}^{\theta}_{\mathfrak{D},\Sigma}$ with $|\mathfrak{D}| \geq \theta$. Then $\mathcal{A} \in \mathrm{Cat}_{\mathfrak{D},\Sigma} \cap \mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma}$. Since $\bot, \top \notin \mathrm{Cat}_{\mathfrak{D},\Sigma}$, it follows that $\mathcal{A}$ is contingent. By Corollary 4.19 we have $|\mathfrak{D}| < \theta$. ⊣

THEOREM 4.6 (essentially Theorem 4.7 of SPLQ). *Let nonempty domain* $\mathfrak{D}$ *and signature* Σ *be given.*

1. *Suppose that* $\mathcal{A} \in \mathrm{Cat}_{\mathfrak{D},\Sigma}$. *Then* $\neg\mathcal{A} \in \mathrm{Sym}_{\mathfrak{D},\Sigma} \setminus \mathrm{Cat}_{\mathfrak{D},\Sigma}$ *if* $|\mathfrak{D}| > 1$.
2. *Suppose that each member of family* $\{\mathcal{A}_\beta\}_{\beta<\gamma}$ *is in* $\mathrm{Cat}_{\mathfrak{D},\Sigma}$ *and that the members of this family are not all logically equivalent one to another. Then both* $\bigvee\{\mathcal{A}_\beta\}_{\beta<\gamma}$ *and* $\bigwedge\{\mathcal{A}_\beta\}_{\beta<\gamma}$ *are members of* $\mathrm{Sym}_{\mathfrak{D},\Sigma} \setminus \mathrm{Cat}_{\mathfrak{D},\Sigma}$.

PROOF. Each of $\neg\mathcal{A}$, $\bigvee\{\mathcal{A}_\beta\}_{\beta<\gamma}$, and $\bigwedge\{\mathcal{A}_\beta\}_{\beta<\gamma}$ is in $\mathrm{Sym}_{\mathfrak{D},\Sigma}$ by Theorem 4.1.

1. First, note that if nonempty $\mathfrak{D}$ has even two elements then $\mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$ is the disjoint union of no fewer than three minimal invariant sets. Suppose, for the sake of contradiction, that both $\mathcal{A}$ and $\neg\mathcal{A}$ are categorical. Then $\mathfrak{T}^{\vec{n}}_{\mathfrak{D}} = \mathrm{Mod}(\mathcal{A}) \cup \mathrm{Mod}(\neg\mathcal{A})$ is the union of just two minimal invariant sets. It follows that $|\mathfrak{D}| = 1$.
2. Let $\mathcal{A}$ and $\mathcal{B}$ be two members of family $\{\mathcal{A}_\beta\}_{\beta<\gamma}$ with $\mathcal{A} \not\Leftrightarrow \mathcal{B}$. That $\mathcal{A}$ is categorical implies that $\emptyset \neq \mathrm{Mod}(\mathcal{A}) \subsetneq \mathrm{Mod}(\mathcal{A}) \dot\cup \mathrm{Mod}(\mathcal{B}) = \mathrm{Mod}(\mathcal{A} \vee \mathcal{B}) \subseteq \mathrm{Mod}(\{\mathcal{A}_\beta\}_{\beta<\gamma})$ since $\mathcal{A}$ and satisfiable $\mathcal{B}$ are inconsistent by Corollary 4.6 of SPLQ. So $\mathrm{Mod}(\bigvee\{\mathcal{A}_\beta\}_{\beta<\gamma})$ has a nonempty proper subset that is invariant, which means that $\bigvee\{\mathcal{A}_\beta\}_{\beta<\gamma})$ is not categorical.

   As for $\bigwedge\{\mathcal{A}_\beta\}_{\beta<\gamma}$, again let $\mathcal{A}$ and $\mathcal{B}$ be two members of family $\{\mathcal{A}_\beta\}_{\beta<\gamma}$ with $\mathcal{A} \not\Leftrightarrow \mathcal{B}$. Then $\mathcal{A}$ and $\mathcal{B}$ are inconsistent by Corollary 4.6 of SPLQ. So $\mathrm{Mod}(\bigwedge\{\mathcal{A}_\beta\}_{\beta<\gamma}) \subseteq \mathrm{Mod}(\mathcal{A} \wedge \mathcal{B}) = \mathrm{Mod}(\mathcal{A}) \cap \mathrm{Mod}(\mathcal{B}) = \emptyset$. Thus $\bigwedge\{\mathcal{A}_\beta\}_{\beta<\gamma}$ is a contradiction and hence not categorical.

⊣

We justify a remark at the very end of §4 of SPLQ.

THEOREM 4.7. *Let domain* $\mathfrak{D}$ *and signature* Σ *be given. Suppose that* $|\mathfrak{D}| < \theta$. *Then the atoms of* $\langle \mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma}/\!\equiv, \bigvee, \bigwedge, \neg, \bot, \top\rangle$ *are all and only the members of* $\mathrm{Cat}^{\theta}_{\mathfrak{D},\Sigma}/\!\equiv$.

PROOF. Suppose that $[\mathcal{A}]_\equiv \in \mathrm{Cat}^{\theta}_{\mathfrak{D},\Sigma}/\!\equiv$ and that $[\mathcal{B}]_\equiv \in \mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma}/\!\equiv$ with $[\mathcal{B}]_\equiv \leq [\mathcal{A}]_\equiv$, which holds just in case $\mathcal{B} \Rightarrow \mathcal{A}$. By Theorem 4.4 either $\mathcal{B}$ is unsatisfiable or $\mathcal{B} \Leftrightarrow \mathcal{A}$ in fact. That is, either $[\mathcal{B}]_\equiv = \bot$ or $[\mathcal{B}]_\equiv = [\mathcal{A}]_\equiv$, which is to say that $[\mathcal{A}]_\equiv$ is an atom.

Suppose that $[\mathcal{A}]_\equiv$ is an atom and let $[\mathcal{B}]_\equiv \in \mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma}/\!\equiv$. Then $[\mathcal{A}]_\equiv \neq \bot$, and $[\mathcal{B}]_\equiv \leq [\mathcal{A}]_\equiv$ implies $[\mathcal{B}]_\equiv = \bot$ or $[\mathcal{B}]_\equiv = [\mathcal{A}]_\equiv$. In other words, $\mathcal{A}$ is satisfiable, and any satisfiable symmetric $\mathcal{B}$ such that $\mathcal{B} \Rightarrow \mathcal{A}$ is such that $\mathcal{B} \Leftrightarrow \mathcal{A}$ in fact. By Theorem 4.4 again we have that $\mathcal{A}$ is categorical so that $[\mathcal{A}]_\equiv \in \mathrm{Cat}^{\theta}_{\mathfrak{D},\Sigma}/\!\equiv$. ⊣

THEOREM 4.8 (see [1] T27-2, pp. 117–18). *Suppose that domain* $\mathfrak{D}$ *and signature* Σ *are given and let* $\tau_1, \tau_2 \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$. *Then we have each of the following.* (We preserve the itemization in [1].)

a. $\mathrm{eldg}(\tau_1) \Rightarrow \bigvee_{\tau\in[\tau_1]_\simeq} \mathrm{eldg}(\tau)$.
b. *If* $\tau_1 \notin [\tau_2]_\simeq$, *then* $\mathrm{eldg}(\tau_1) \Rightarrow \neg\bigvee_{\tau\in[\tau_2]_\simeq} \mathrm{eldg}(\tau)$.
c. *If* $\mathrm{eldg}(\tau_1) \Rightarrow \bigvee_{\tau\in[\tau_2]_\simeq} \mathrm{eldg}(\tau)$, *then* $\tau_1 \in [\tau_2]_\simeq$.
d. $\tau_1 \in \mathrm{Mod}(\bigvee_{\tau\in[\tau_1]_\simeq} \mathrm{eldg}(\tau))$.
e. *If* $\tau_1 \in \mathrm{Mod}(\bigvee_{\tau\in[\tau_2]_\simeq} \mathrm{eldg}(\tau))$, *then* $\tau_1 \in [\tau_2]_\simeq$.

f. *The following four conditions are equivalent:*
   1. $\tau_1 \in [\tau_2]_{\simeq}$.
   2. $\pi(\tau_1) = \tau_2$ *for some* $\pi \in S_{\mathfrak{D}}$.
   5. $\mathrm{eldg}(\tau_1) \Rightarrow \bigvee_{\tau \in [\tau_2]_{\simeq}} \mathrm{eldg}(\tau)$.
   6. $\tau_1 \in \mathrm{Mod}(\bigvee_{\tau \in [\tau_2]_{\simeq}} \mathrm{eldg}(\tau))$.

THEOREM 4.9. *Suppose that domain* $\mathfrak{D}$ *and signature* Σ *are given and let* $\tau_1, \tau_2 \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$. *Then we have each of the following.* (Itemization follows that of Theorem 4.8.)

a. *There exists* $\mathcal{A} \in \mathrm{Cat}_{\mathfrak{D},\Sigma}$ *such that* $\mathrm{eldg}(\tau_1) \Rightarrow \mathcal{A}$.
b. *There exists* $\mathcal{A} \in \mathrm{Cat}_{\mathfrak{D},\Sigma}$ *such that* $\mathrm{eldg}(\tau_1) \Rightarrow \neg\mathcal{A}$.
c. *If* $\mathrm{eldg}(\tau_1) \Rightarrow \mathcal{A}$ *with* $\mathcal{A} \in \mathrm{Cat}_{\mathfrak{D},\Sigma}$ *and* $\tau_2 \in \mathrm{Mod}(\mathcal{A})$, *then* $\pi(\tau_1) = \tau_2$ *for some* $\pi \in S_{\mathfrak{D}}$.
d. *There exists* $\mathcal{A} \in \mathrm{Cat}_{\mathfrak{D},\Sigma}$ *such that* $\tau_1 \in \mathrm{Mod}(\mathcal{A})$.
e. *If* $\tau_1, \tau_2 \in \mathrm{Mod}(\mathcal{A})$ *with* $\mathcal{A} \in \mathrm{Cat}_{\mathfrak{D},\Sigma}$, *then* $\pi(\tau_1) = \tau_2$ *for some* $\pi \in S_{\mathfrak{D}}$.
f. *The following four conditions are equivalent:*
   1. $\tau_1 \in [\tau_2]_{\simeq}$.
   2. $\pi(\tau_1) = \tau_2$ *for some* $\pi \in S_{\mathfrak{D}}$.
   5. $\mathrm{eldg}(\tau_1) \Rightarrow \mathcal{A}$ *for some* $\mathcal{A} \in \mathrm{Cat}_{\mathfrak{D},\Sigma}$ *with* $\tau_2 \in \mathrm{Mod}(\mathcal{A})$.
   6. $\tau_1, \tau_2 \in \mathrm{Mod}(\mathcal{A})$ *for some* $\mathcal{A} \in \mathrm{Cat}_{\mathfrak{D},\Sigma}$.

4.1. Symmetry, contingency, and domain size. As mentioned following Definition 4.1 of SPLQ, one can prove that $\mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma}$ has a contingent member only if $|\mathfrak{D}| < \theta$. The claim plays a role at the very end of §4 of SPLQ. Also, our proof of First Representation Theorem 6.1 makes use of it. We establish the claim in Corollary 4.19 below. Considerable complications arise along the way due to the fact that, if $\mathfrak{D}$ is finite, then the number of nonconstituents of proposition $\mathcal{A} \in \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ with $\Sigma = \langle R_1^{n_1}, \ldots, R_p^{n_p} \rangle$ may be strictly less than n_ℓ for some $1 \le \ell \le p$.

DEFINITION 4.10. *Let* $\mathcal{A}$ *be a member of system* $\mathcal{H}_{\mathfrak{D},\Sigma}$. *Then* $\mathrm{Constit}(\mathcal{A}) \subseteq \mathfrak{D}$ *is defined inductively as follows.*

1. *If* $\mathcal{A}$ *is elementary* $R_\ell \mathfrak{a}_1 \ldots \mathfrak{a}_{n_\ell}$, *then* $\mathrm{Constit}(\mathcal{A}) = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_{n_\ell}\}$.
2. *If* $\mathcal{A}$ *is* $\neg(\mathcal{B})$, *then* $\mathrm{Constit}(\mathcal{A}) = \mathrm{Constit}(\mathcal{B})$.
3. *If* $\mathcal{A}$ *is* $\bigvee K$ *with* $K \subseteq \mathcal{H}_{\mathfrak{D},\Sigma}$, *then* $\mathrm{Constit}(\mathcal{A}) = \bigcup_{\mathcal{B} \in K} \mathrm{Constit}(\mathcal{B})$.

THEOREM 4.11. *For any* $\mathcal{A} \in \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ *we have* $|\mathrm{Constit}(\mathcal{A})| < \theta$.

PROOF. By induction on the rank of proposition $\mathcal{A}$. ⊣

LEMMA 4.12. *Let* $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},\Sigma}$ *and suppose that* $\mathfrak{b} \in \mathfrak{D} \setminus \mathrm{Constit}(\mathcal{A})$. *Let* $\tau =: \langle f_1, \ldots, f_p \rangle \in \mathrm{Mod}(\mathcal{A})$. *Fix* ℓ *with* $1 \le \ell \le p$ *and* j *with* $1 \le j \le n_\ell$. *Let* $\mathfrak{a}_1, \ldots, \mathfrak{a}_{j-1}, \mathfrak{a}_{j+1}, \ldots, \mathfrak{a}_{n_\ell} \in \mathfrak{D}$. *Finally, let* $\tau' =: \langle f'_1, \ldots, f'_p \rangle$ *be the unique member of* $\mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$ *satisfying*

1. *if* $k \neq \ell$, *then* $f'_k = f_k$
2. $f'_\ell(\mathfrak{a}_1, \ldots, \mathfrak{a}_{j-1}, \mathfrak{b}, \mathfrak{a}_{j+1}, \ldots, \mathfrak{a}_{n_\ell}) \neq f_\ell(\mathfrak{a}_1, \ldots, \mathfrak{a}_{j-1}, \mathfrak{b}, \mathfrak{a}_{j+1}, \ldots, \mathfrak{a}_{n_\ell})$
3. *for any* $\langle \mathfrak{b}_1, \ldots, \mathfrak{b}_{n_\ell} \rangle \in \mathfrak{D}^{n_\ell}$ *distinct from* $\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_{j-1}, \mathfrak{b}, \mathfrak{a}_{j+1}, \ldots, \mathfrak{a}_{n_\ell} \rangle$, *we have* $f'_\ell(\mathfrak{b}_1, \ldots, \mathfrak{b}_{n_\ell}) = f_\ell(\mathfrak{b}_1, \ldots, \mathfrak{b}_{n_\ell})$.

Then $\tau' \in \mathrm{Mod}(\mathcal{A})$.

Proof. If $\mathcal{A}$ is a contradiction, then the claim is vacuously true. If a tautology, then it is equally obvious. We proceed by induction on the complexity of $\mathcal{A}$ and assume that $\emptyset \subsetneq \text{Mod}(\mathcal{A}) \subsetneq \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$. If $\mathcal{A}$ is elementary $R_\ell \mathfrak{a}_1 \ldots \mathfrak{a}_{n_\ell}$, then the claim is obvious. Suppose that $\mathcal{A}$ is $\neg(\mathcal{B})$, that $\mathfrak{b}$ is not among its constituents, and that $\tau =: \langle f_1, \ldots, f_p \rangle \in \text{Mod}(\mathcal{A})$. Consider $\tau' =: \langle f'_1, \ldots, f'_p \rangle$ satisfying conditions (1)–(3). If $\tau' \in \text{Mod}(\mathcal{B})$, then by induction hypothesis so is τ since $\mathfrak{b}$ is not a constituent of $\mathcal{B}$. But this contradicts $\tau \in \text{Mod}(\neg(\mathcal{B}))$. So $\tau' \notin \text{Mod}(\mathcal{B})$, in which case $\tau' \in \text{Mod}(\neg(\mathcal{B}))$, which is $\text{Mod}(\mathcal{A})$.

Suppose that $\mathcal{A}$ is $\bigvee K$ with $K \subseteq \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$, that $\mathfrak{b}$ is not among its constituents, and that $\tau =: \langle f_1, \ldots, f_p \rangle \in \text{Mod}(\mathcal{A}) = \bigcup_{\mathcal{B} \in K} \text{Mod}(\mathcal{B})$. Let $\tau' =: \langle f'_1, \ldots, f'_p \rangle$ satisfy conditions (1)–(3). Then $\tau \in \text{Mod}(\mathcal{B})$ for some $\mathcal{B}$, and, since $\mathfrak{b} \notin \text{Constit}(\mathcal{B})$, we have $\tau' \in \text{Mod}(\mathcal{B}) \subseteq \text{Mod}(\mathcal{A})$ by induction hypothesis. ⊣

Let $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},\Sigma}$. If $\tau, \tau' \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$ differ on nonconstituents of $\mathcal{A}$ only, then either $\tau, \tau' \in \text{Mod}(\mathcal{A})$ or $\tau, \tau' \notin \text{Mod}(\mathcal{A})$, as we now show.

Theorem 4.13. *Let* $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},\Sigma}$ *and suppose that* $\tau =: \langle f_1, \ldots, f_p \rangle \in \text{Mod}(\mathcal{A})$. *Let* $\tau' =: \langle f'_1, \ldots, f'_p \rangle \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$. *Now suppose that, for any* $1 \leq \ell \leq p$ *and any* $\mathfrak{b}_1, \ldots, \mathfrak{b}_{n_\ell} \in \mathfrak{D}$, *we have that* $f_\ell(\mathfrak{b}_1, \ldots, \mathfrak{b}_{n_\ell}) \neq f'_\ell(\mathfrak{b}_1, \ldots, \mathfrak{b}_{n_\ell})$ *implies* $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_{n_\ell}\} \not\subseteq \text{Constit}(\mathcal{A})$. *Then* $\tau' \in \text{Mod}(\mathcal{A})$.

Proof. By iterated application of Lemma 4.12. ⊣

Remarks 4.14 and 4.15 follow immediately.

Remark 4.14. If $\text{Mod}(\mathcal{A})$ contains τ but not τ', then τ and τ' must differ on some n_ℓ-tuple consisting of constituents of $\mathcal{A}$.

Remark 4.15. Suppose that $\text{Mod}(\mathcal{A})$ contains τ but not τ'. We may assume without loss of generality that such τ and τ' agree on all nonconstituents. That is, we may assume that, for any $1 \leq \ell \leq p$ and any $\mathfrak{b}_1, \ldots, \mathfrak{b}_{n_\ell} \in \mathfrak{D} \setminus \text{Constit}(\mathcal{A})$, we have $f_\ell(\mathfrak{b}_1, \ldots, \mathfrak{b}_{n_\ell}) = f'_\ell(\mathfrak{b}_1, \ldots, \mathfrak{b}_{n_\ell})$.

Where $\mathfrak{D} \subseteq \mathfrak{D}'$, we write $S_{\mathfrak{D}}$ and $S_{\mathfrak{D}'}$ for the classes of permutations of $\mathfrak{D}$ and $\mathfrak{D}'$, respectively. With $\pi \in S_{\mathfrak{D}'}$ we write $\pi \restriction \mathfrak{D} =: \{\langle \mathfrak{a}, \mathfrak{b} \rangle \in \mathfrak{D} \times \mathfrak{D} | \pi(\mathfrak{a}) = \mathfrak{b}\}$ for a certain member of $S_{\mathfrak{D}}$ provided that π fixes $\mathfrak{D}' \setminus \mathfrak{D}$ setwise. Of course, for all $\tau \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}'}$ and all $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},\Sigma}$, we have

$$(\star) \qquad \tau \in \text{Mod}_{\mathfrak{D}',\Sigma}(\mathcal{A}) \text{ if and only if } \tau \restriction \mathfrak{D} \in \text{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A}).$$

Also, if $\pi \in S_{\mathfrak{D}'}$ fixes $\mathfrak{D}$ setwise, then, for all $\tau \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}'}$ we have that

$$(\star\star) \qquad \pi(\tau) \restriction \mathfrak{D} = \pi \restriction \mathfrak{D}(\tau \restriction \mathfrak{D}).$$

Further, if $\pi \in S_{\mathfrak{D}'}$ fixes $\mathfrak{D}$ setwise, then $(\star)$ and $(\star\star)$ together give

$$(+) \qquad \pi(\tau) \in \text{Mod}_{\mathfrak{D}',\Sigma}(\mathcal{A}) \text{ if and only if } \pi \restriction \mathfrak{D}(\tau \restriction \mathfrak{D}) \in \text{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A})$$

for all $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},\Sigma}$. As a special case, let $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},\Sigma}$ and suppose that π fixes $\text{Constit}(A) \subseteq \mathfrak{D}$ setwise. Then (+) becomes

$$(\dagger)\ \pi(\tau) \in \text{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A}) \text{ if and only if}$$
$$\pi \restriction \text{Constit}(\mathcal{A})(\tau \restriction \text{Constit}(\mathcal{A})) \in \text{Mod}_{\text{Constit}(\mathcal{A}),\Sigma}(\mathcal{A}).$$

The following theorem will be used in the proof of Theorem 4.18.

THEOREM 4.16. *Suppose that* $\mathcal{A} \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$ *but that* $\mathrm{Constit}(\mathcal{A}) \subsetneq \mathfrak{D}$. *Suppose that* $\mathfrak{D} \subsetneq \mathfrak{D}' = \mathfrak{D} \cup \{\mathfrak{a}_1, \ldots, \mathfrak{a}_k\}$. *Then* $\mathcal{A} \in \mathrm{Sym}_{\mathfrak{D}',\Sigma}$ *also.*

PROOF. It is convenient to assume that $\mathfrak{D}' = \mathfrak{D} \cup \{a\}$ initially. (For the general case whereby $\mathfrak{D}' = \mathfrak{D} \cup \{\mathfrak{a}_1, \ldots, \mathfrak{a}_k\}$, we iterate the ensuing argument k times.) For the sake of contradiction, assume $\mathcal{A} \notin \mathrm{Sym}_{\mathfrak{D}',\Sigma}$. Since $\mathcal{A}$ is not symmetric over $\mathfrak{D}'$ and Σ, some permutation π of $\mathfrak{D}'$ takes $\tau =: \langle f_1, \ldots, f_p \rangle \in \mathrm{Mod}_{\mathfrak{D}',\Sigma}(\mathcal{A})$ to $\pi(\tau) = \langle \pi(f_1), \ldots, \pi(f_p) \rangle \notin \mathrm{Mod}_{\mathfrak{D}',\Sigma}(\mathcal{A})$. We have $\tau \upharpoonright \mathfrak{D} =: \langle f_1 \upharpoonright \mathfrak{D}, \ldots, f_p \upharpoonright \mathfrak{D} \rangle \in \mathrm{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A})$ since omitted a is a nonconstituent. (We forego the easy proof by induction on the complexity of $\mathcal{A}$.) Similarly, $\pi(\tau) \upharpoonright \mathfrak{D} =: \langle \pi(f_1) \upharpoonright \mathfrak{D}, \ldots, \pi(f_p) \upharpoonright \mathfrak{D} \rangle \notin \mathrm{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A})$. There are four cases to consider.

1. Suppose that π fixes a. We have $\pi \upharpoonright \mathfrak{D}(\tau \upharpoonright \mathfrak{D}) = \pi(\tau) \upharpoonright \mathfrak{D}$ since π fixes a. But then $\pi \upharpoonright \mathfrak{D}(\tau \upharpoonright \mathfrak{D}) \notin \mathrm{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A})$ by an earlier remark, contradicting $\mathcal{A} \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$.
2. Suppose that π does not fix a but does fix some $b \in \mathfrak{D} \setminus \mathrm{Constit}(\mathcal{A}) \neq \emptyset$. We define $\pi' \in S_{\mathfrak{D}'}$ by writing

$$\pi'(\mathfrak{c}) = \begin{cases} \pi(\mathfrak{c}) & \text{if } \mathfrak{c} \notin \{\pi^{-1}(a), a, b\} \\ b & \text{if } \mathfrak{c} = \pi^{-1}(a) \\ a & \text{if } \mathfrak{c} = a \\ \pi(a) & \text{if } \mathfrak{c} = b. \end{cases}$$

   Since π' fixes a, we have $\pi' \upharpoonright \mathfrak{D}(\tau \upharpoonright \mathfrak{D}) \notin \mathrm{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A})$ by (1) with $\pi' \upharpoonright \mathfrak{D} \in S_{\mathfrak{D}}$. This contradicts $\mathcal{A} \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$.
3. Suppose that π fixes no element of $\mathfrak{D} \setminus \mathrm{Constit}(\mathcal{A})) \cup \{a\}$ but that π does fix $(\mathfrak{D} \setminus \mathrm{Constit}(\mathcal{A})) \cup \{a\}$ setwise. It follows that π fixes $\mathrm{Constit}(\mathcal{A})$ setwise, and $\pi(\tau) \notin \mathrm{Mod}_{\mathfrak{D}',\Sigma}(\mathcal{A})$ implies $\pi \upharpoonright \mathrm{Constit}(\mathcal{A})(\tau \upharpoonright \mathrm{Constit}(\mathcal{A})) \notin \mathrm{Mod}_{\mathrm{Constit}(\mathcal{A}),\Sigma}(\mathcal{A})$ by (+) and $\mathcal{A} \in \mathcal{H}_{\mathrm{Constit}(\mathcal{A}),\Sigma}$. Let $\pi' \in S_{\mathfrak{D}}$ be defined by

$$\pi'(\mathfrak{a}) = \begin{cases} [\pi \upharpoonright \mathrm{Constit}(\mathcal{A})](\mathfrak{a}) & \text{if } \mathfrak{a} \in \mathrm{Constit}(\mathcal{A}) \\ \mathfrak{a} & \text{if } \mathfrak{a} \in \mathfrak{D} \setminus \mathrm{Constit}(\mathcal{A}). \end{cases}$$

   Then $\pi \upharpoonright \mathrm{Constit}(A)(\tau \upharpoonright \mathrm{Constit}(\mathcal{A})) \notin \mathrm{Mod}_{\mathrm{Constit}(\mathcal{A}),\Sigma}(\mathcal{A})$ has as consequence that $\pi'(\tau \upharpoonright \mathfrak{D}) \notin \mathrm{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A})$. (Recall (†) and use $\pi' \upharpoonright \mathrm{Constit}(A) = \pi \upharpoonright \mathrm{Constit}(\mathcal{A})$ as well as $(\tau \upharpoonright \mathfrak{D}) \upharpoonright \mathrm{Constit}(\mathcal{A}) = \tau \upharpoonright \mathrm{Constit}(\mathcal{A})$.) This contradicts $\mathcal{A} \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$.
4. Finally, assume that π does not fix $(\mathfrak{D}' \setminus \mathrm{Constit}(\mathcal{A})) \cup \{a\}$ setwise. It is convenient to assume initially that π maps just one nonconstituent to a constituent, and vice versa. That is, π transposes nonconstituent a_{nc} and constituent a_{c} and otherwise fixes both $[(\mathfrak{D}' \setminus \mathrm{Constit}(\mathcal{A})) \cup \{a\}] \setminus \{a_{\mathrm{nc}}\}$ and $\mathrm{Constit}(\mathcal{A}) \setminus \{a_{\mathrm{c}}\}$.

(a) As first subcase, suppose that $a_{\mathrm{nc}} \in \mathfrak{D}$ so that $a_{\mathrm{nc}} \neq a$. We define $\pi' \in S_{\mathfrak{D}'}$ by writing

$$\pi'(\mathfrak{c}) = \begin{cases} \pi(\mathfrak{c}) & \text{if } \mathfrak{c} \notin \{\pi^{-1}(a), a, \pi(a)\} \\ \pi(a) & \text{if } \mathfrak{c} = \pi^{-1}(a) \\ a & \text{if } \mathfrak{c} = a \\ \pi^{-1}(a) & \text{if } \mathfrak{c} = \pi(a). \end{cases}$$

Since π' differs from π on nonconstituents only, it is clear that $\pi(\tau) \notin \mathrm{Mod}_{\mathfrak{D}',\Sigma}(\mathcal{A})$ implies $\pi'(\tau) \notin \mathrm{Mod}_{\mathfrak{D}',\Sigma}(\mathcal{A})$. (We forego an easy proof by induction on the complexity of $\mathcal{A}$.) And, again, since π' fixes a, we have $\pi' \restriction \mathfrak{D}(\tau \restriction \mathfrak{D}) \notin \mathrm{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A})$ by (1) with $\pi' \restriction \mathfrak{D} \in S_{\mathfrak{D}}$. This contradicts $\mathcal{A} \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$.

(b) Otherwise, assume that $a_{\mathrm{nc}} = a \in \mathfrak{D}' \setminus \mathfrak{D}$. Let $a' \in \mathfrak{D} \setminus \mathrm{Constit}(\mathcal{A}) \neq \emptyset$. By Lemma 4.17 there exists $\tau' \in \mathrm{Mod}_{\mathfrak{D},\Sigma}$ such that the role of a' in τ' is precisely that of a_{nc} in τ (cf. condition (C) in the statement of Lemma 4.17). Let $\pi' \in S_{\mathfrak{D}'}$ be defined by

$$\pi'(\mathfrak{c}) = \begin{cases} \pi(\mathfrak{c}) & \text{if } \mathfrak{c} \notin \{\pi^{-1}(a'), a', a = a_{\mathrm{nc}}, a_{\mathrm{c}}\} \\ \pi(a') & \text{if } \mathfrak{c} = \pi^{-1}(a') \\ a_{\mathrm{c}} & \text{if } \mathfrak{c} = a' \\ a & \text{if } \mathfrak{c} = a = a_{\mathrm{nc}} \\ a' & \text{if } \mathfrak{c} = a_{\mathrm{c}}. \end{cases}$$

Since $\pi(\tau) \notin \mathrm{Mod}_{\mathfrak{D}',\Sigma}$ by assumption, (C) gives $\pi'(\tau') \notin \mathrm{Mod}_{\mathfrak{D}',\Sigma}$.[1] But π' fixes a so that $\pi' \restriction \mathfrak{D}(\tau \restriction \mathfrak{D}) \notin \mathrm{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A})$ by (+) with $\pi' \restriction \mathfrak{D} \in S_{\mathfrak{D}}$. This contradicts $\mathcal{A} \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$.

If, in contrast, π maps exactly two nonconstituents a_{nc_1} and a_{nc_2} to constituents a_{c_1} and a_{c_2}, respectively, then again two subcases arise. The case whereby both of a_{nc_1} and a_{nc_2} are in $\mathfrak{D}$ can be reduced to subcase (4a). Otherwise, there is a reduction to subcase (4b). Continuing in this manner, we conclude that a contradiction arises even if π should take each and every nonconstituent to a constituent.

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[1] We show that, where $\tau =: \langle f_1, \ldots, f_p \rangle \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$, is arbitrary, $\tau' =: \langle f'_1, \ldots, f'_p \rangle \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$, satisfies condition (C), and $\pi, \pi' \in S_{\mathfrak{D}'}$ are as described above, we have $\pi(\tau) \in \mathrm{Mod}_{\mathfrak{D}',\Sigma}(\mathcal{A})$ if and only if $\pi'(\tau') \in \mathrm{Mod}_{\mathfrak{D}',\Sigma}(\mathcal{A})$. For the base case, suppose that $\mathcal{A}$ is $R_\ell \mathfrak{a}_1 \ldots \mathfrak{a}_{n_\ell}$ so that $\mathfrak{a}_1, \ldots, \mathfrak{a}_{n_\ell} \in \mathrm{Constit}(\mathcal{A})$. Then

$$\begin{aligned} &\pi(\tau) \in \mathrm{Mod}_{\mathfrak{D}',\Sigma}(\mathcal{A}) \\ \Longleftrightarrow &\pi(f_\ell)(\mathfrak{a}_1, \ldots, \mathfrak{a}_{n_\ell}) = 1 \\ \Longleftrightarrow &f_\ell(\pi^{-1}(\mathfrak{a}_1), \ldots, \pi^{-1}(\mathfrak{a}_{n_\ell})) = 1 \\ \Longleftrightarrow &f_\ell([\pi']^{-1}(\mathfrak{a}_1), \ldots, [\pi']^{-1}(\mathfrak{a}_{n_\ell})) = 1 \end{aligned}$$

by the definition of τ' since for all $1 \leq i \leq n_\ell$ either (1) $\mathfrak{a}_i = a_{\mathrm{c}}$, in which case $\pi^{-1}(\mathfrak{a}_i) = a_{\mathrm{nc}} = a$ and $[\pi']^{-1}(\mathfrak{a}_i) = a'$ or (ii) $\mathfrak{a}_i = a_{\mathrm{c}}$, in which case $[\pi']^{-1}(\mathfrak{a}_i) = \pi^{-1}(\mathfrak{a}_i)$

$$\begin{aligned} \Longleftrightarrow &\pi'(f'_\ell)(\mathfrak{a}_1, \ldots, \mathfrak{a}_{n_\ell}) = 1 \\ \Longleftrightarrow &\pi'(\tau') \in \mathrm{Mod}_{\mathfrak{D}',\Sigma}(\mathcal{A}). \end{aligned}$$

Two inductive cases are straightforward.

⊣

Lemma 4.17. *Let* $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},\Sigma}$ *with* $|\mathfrak{D} \setminus \mathrm{Constit}(\mathcal{A})| > 1$ *and suppose that* $\tau =: \langle f_1, \dots, f_p \rangle \in \mathrm{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A})$. *Let* $a, a' \in \mathfrak{D} \setminus \mathrm{Constit}(\mathcal{A})$ *with* $a \neq a'$. *Then there exists* $\tau' =: \langle f'_1, \dots, f'_p \rangle \in \mathrm{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A})$ *satisfying the following condition.*

(C) *For all* $1 \leq \ell \leq p$, *we have that* $f_\ell(\mathfrak{a}_1, \dots, \mathfrak{a}_{n_\ell}) = f'_\ell(\mathfrak{b}_1, \dots, \mathfrak{b}_{n_\ell})$ *whenever (i)* $\{\mathfrak{a}_1, \dots, \mathfrak{a}_{n_\ell}\} \cap \{a\} \neq \emptyset$, *(ii) for all* $1 \leq i \leq n_\ell$ *we have that* $\mathfrak{a}_i = a$ *if and only if* $\mathfrak{b}_i = a'$, *and (iii) for all* $1 \leq i \leq n_\ell$ *we have that* $\mathfrak{a}_i \neq a$ *implies* $\mathfrak{a}_i = \mathfrak{b}_i$. *("The role of nonconstituent* a *within* τ *is identical with that of nonconstituent* a' *within* τ'*.")*

Proof. Suppose that $\tau \in \mathrm{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A})$. For all $1 \leq \ell \leq p$ and all $\mathfrak{a}_1, \dots, \mathfrak{a}_{n_\ell} \in \mathrm{Constit}(\mathcal{A})$, let $\tau' \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$ satisfy $f'_\ell(\mathfrak{a}_1, \dots, \mathfrak{a}_{n_\ell}) = f_\ell(\mathfrak{a}_1, \dots, \mathfrak{a}_{n_\ell})$. It follows that $\tau' \in \mathrm{Mod}_{\mathfrak{D},\Sigma}(\mathcal{A})$. Since $\mathrm{Constit}(\mathcal{A}) \cap \{a\} = \emptyset$ so that (i) fails for any n_ℓ-tuple involving constituents only, we can assume without loss of generality that τ' satisfies condition (C) as well. ⊣

We are now ready to present our main result.

Theorem 4.18. *If* $\mathcal{A} \in \mathrm{Sym}^\theta_{\mathfrak{D},\Sigma}$ *is contingent, then* $\mathrm{Constit}(\mathcal{A}) = \mathfrak{D}$.

Proof. Since $\mathcal{A}$ is contingent, we have $\emptyset \subsetneq \mathrm{Mod}(\mathcal{A}) \subsetneq \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$. Suppose, for the sake of contradiction, that $\mathfrak{D} \setminus \mathrm{Constit}(\mathcal{A}) \neq \emptyset$. By Theorem 4.16 we may assume without loss of generality that $|\mathfrak{D} \setminus \mathrm{Constit}(\mathcal{A})| \geq n_\ell$ for all $1 \leq \ell \leq p$. (If not, we add new elements to $\mathfrak{D}$.) Let $\tau =: \langle f_1, \dots, f_p \rangle \in \mathrm{Mod}(\mathcal{A})$ and let $\tau' =: \langle f'_1, \dots, f'_p \rangle \notin \mathrm{Mod}(\mathcal{A})$. By Remark 4.15, we may assume that τ and τ' agree on all n_ℓ-tuples involving nonconstituents only. By symmetry we have $\pi^{-1}(\tau) = \langle \pi^{-1}(f_1), \dots, \pi^{-1}(f_p) \rangle \in \mathrm{Mod}(\mathcal{A})$ for arbitrary $\pi \in S_{\mathfrak{D}}$ whereas $\pi^{-1}(\tau') = \langle \pi^{-1}(f'_1), \dots, \pi^{-1}(f'_p) \rangle \notin \mathrm{Mod}(\mathcal{A})$. By Remark 4.14 we have that $[\pi^{-1}(f_k)](\vec{\mathfrak{b}}) \neq [\pi^{-1}(f'_k)](\vec{\mathfrak{b}})$ for some $1 \leq k \leq p$ and some $\mathfrak{b}_1, \dots, \mathfrak{b}_{n_k}$ satisfying $\{\mathfrak{b}_1, \dots, \mathfrak{b}_{n_k}\} \subseteq \mathrm{Constit}(\mathcal{A})$. (We do not assume that $\mathfrak{b}_1, \dots, \mathfrak{b}_{n_k}$ are distinct.) Let $\tilde{\pi} \in S_{\mathfrak{D}}$ satisfy $\tilde{\pi}(\mathfrak{b}_1) = \mathfrak{a}_1, \dots, \tilde{\pi}(\mathfrak{b}_{n_k}) = \mathfrak{a}_{n_k}$, where $\mathfrak{a}_1, \dots, \mathfrak{a}_{n_k} \in \mathfrak{D} \setminus \mathrm{Constit}(\mathcal{A})$. (Thus $\mathfrak{a}_1, \dots, \mathfrak{a}_{n_k}$ may not be distinct.) But then

$$f_k(\vec{\mathfrak{a}}) = f_k(\tilde{\pi}(\vec{\mathfrak{b}})) = [\tilde{\pi}^{-1}(f_k)](\vec{\mathfrak{b}}) \neq [\tilde{\pi}^{-1}(f'_k)](\vec{\mathfrak{b}}) = f'_k(\tilde{\pi}(\vec{\mathfrak{b}})) = f'_k(\vec{\mathfrak{a}}).$$

Thus τ and τ' disagree on nonconstituents $\mathfrak{a}_1, \dots, \mathfrak{a}_{n_k}$ after all, which is a contradiction. ⊣

Corollary 4.19. *If* $\mathrm{Sym}^\theta_{\mathfrak{D},\Sigma}$ *has a contingent member, then* $|\mathfrak{D}| < \theta$.

Proof. An easy inductive argument shows that the constituents of any member $\mathcal{A}$ of $\mathcal{H}^\theta_{\mathfrak{D},\Sigma}$ number fewer than θ. If $\mathcal{A}$ happens to be both contingent and symmetric, then by Theorem 4.18 its constituent set coincides with $\mathfrak{D}$. ⊣

4.2. Classes versus sets. We use the terms "class" and "set" advisedly with respect to Zermelo systems and Zermelo logics, as we now explain. In the minimal case, we have $\mathfrak{D} = \{a\}$ and unit signature $\Sigma = \langle R \rangle$ with R monadic. Fundamental relation Ra is of rank 1, $\neg(Ra)$ is of rank 2, $\neg(\neg(Ra))$ is of rank 3, and so forth. Writing $\neg^0(Ra)$, $\neg^1(Ra)$, and $\neg_2(Ra)$ for the three propositions just mentioned, we have that $\bigvee_{n<\omega} \neg^n(Ra)$ is of rank $\omega + 1$. (By Lemma 2.1, no proposition has limit rank.[2]) Thus, for all $\alpha \in \mathrm{On}$, some proposition of Zermelo system $\mathcal{H}_{\mathfrak{D},\Sigma}$ is of rank $\alpha+1$. Further, every proposition has ordinal rank. We conclude that the cardinality of $\mathcal{H}_{\mathfrak{D},\Sigma}$ is that of On. In consequence of this, we say that $\mathcal{H}_{\mathfrak{D},\Sigma}$ is a *class*. In this minimal setting Ra, $\neg(Ra)$, $\neg(\neg(Ra))$, … are all categorical. So are $\bigvee\{\neg^n(Ra) \mid n \text{ even}\}$ and $\bigvee\{\neg^n(Ra) \mid n \text{ odd}\}$. So $|\mathrm{Cat}_{\mathfrak{D},\Sigma}| = |\mathcal{H}_{\mathfrak{D},\Sigma}|$, which means that $\mathrm{Cat}_{\mathfrak{D},\Sigma}$ is a class, as is $\mathrm{Sym}_{\mathfrak{D},\Sigma} \supseteq \mathrm{Cat}_{\mathfrak{D},\Sigma}$.

Turning to Zermelo logics, we let some strongly inaccessible θ play the role of On in the previous paragraph. Since $|\mathcal{H}^\theta_{\mathfrak{D},\Sigma}| = \theta$, as is easily seen, we say that logic $\mathcal{H}^\theta_{\mathfrak{D},\Sigma}$ is a class. (This is true for arbitrary nonempty domain $\mathfrak{D}$ and arbitrary nonempty signature Σ.) Every tautology of logic $\mathcal{H}^\theta_{\mathfrak{D},\Sigma}$ is symmetric, and the tautologies of $\mathcal{H}^\theta_{\mathfrak{D},\Sigma}$ number θ. (In the minimal setting, each of $\neg^n(Ra \vee \neg(Ra))$ for even n is tautologous, as is each of $\neg^m(\bigvee\{\neg^n(Ra \vee \neg(Ra)) \mid n \text{ even}\})$ for even m, and so forth.) So $\mathrm{Sym}^\theta_{\mathfrak{D},\Sigma}$ is a class. As for $\mathrm{Cat}^\theta_{\mathfrak{D},\Sigma}$, matters are more complex. It is empty if $|\mathfrak{D}| \geq \theta$ and hence a set. If $|\mathfrak{D}| < \theta$, then $|\mathrm{Cat}^\theta_{\mathfrak{D},\Sigma}| = \theta$. (In the minimal setting each of $\neg^n(Ra)$ for $n \geq 0$ is categorical, as is each of $\neg^m(\bigvee\{\neg^n(Ra) \mid n \text{ even}\})$ for $m \geq 0$, and so forth.) So if $|\mathfrak{D}| < \theta$, then $\mathrm{Cat}^\theta_{\mathfrak{D},\Sigma}$ is a class. For simplicity we say generally that $\mathrm{Cat}^\theta_{\mathfrak{D},\Sigma}$ is a class.

§5. Zermelo logics and quantifier algebras. Where $n_1, \ldots, n_p \geq 0$, we have that

$$\left\langle \mathcal{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle, \curlyvee, \curlywedge, \sim, \mathsf{false}^{\vec{n}}, \mathsf{true}^{\vec{n}} \right\rangle$$

is a θ-complete atomic Boolean algebra of quantifiers over $\mathfrak{D}$, where operators and constants are defined by the following:

$1_{\mathfrak{D}}$. Where $\mathcal{K} \subseteq \mathcal{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ satisfies $|\mathcal{K}| < \theta$, type-$\langle\vec{n}\rangle$ quantifier $\curlyvee\mathcal{K}$ maps p-tuple $\langle A_1, \ldots, A_p\rangle \in \wp(\mathfrak{D}^{n_1}) \times \ldots \times \wp(\mathfrak{D}^{n_p})$ to true just in case we have that $Q^{\vec{n}}(A_1, \ldots, A_p) = \text{true}$ for some $Q^{\vec{n}} \in \mathcal{K}$.

$2_{\mathfrak{D}}$. Where $\mathcal{K} \subseteq \mathcal{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ satisfies $|\mathcal{K}| < \theta$, type-$\langle\vec{n}\rangle$ quantifier $\curlywedge\mathcal{K}$ maps p-tuple $\langle A_1, \ldots, A_p\rangle \in \wp(\mathfrak{D}^{n_1}) \times \ldots \times \wp(\mathfrak{D}^{n_p})$ to true just in case we have that $Q^{\vec{n}}(A_1, \ldots, A_p) = \text{true}$ for every $Q^{\vec{n}} \in \mathcal{K}$.

$3_{\mathfrak{D}}$. Given $Q^{\vec{n}} \in \mathcal{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$, type-$\langle\vec{n}\rangle$ quantifier $\sim Q^{\vec{n}}$ maps p-tuple $\langle A_1, \ldots, A_p\rangle \in \wp(\mathfrak{D}^{n_1}) \times \ldots \times \wp(\mathfrak{D}^{n_p})$ to true just in case $Q^{\vec{n}}$ maps $\langle A_1, \ldots, A_p\rangle$ to false.

$4_{\mathfrak{D}}$. Type-$\langle\vec{n}\rangle$ quantifiers $\mathsf{false}^{\vec{n}}$ and $\mathsf{true}^{\vec{n}}$ are the constant-false and constant-true function, respectively, on $\wp(\mathfrak{D}^{j_1}) \times \wp(\mathfrak{D}^{j_2})$.

Our notion of montagovian $\langle n_\ell\rangle$-tuple is a generalization of the notion introduced in R. Montague's intensional logic, whereby (1) the intension of proper noun **John** is identified with a class of properties whereas (2) the extension (or denotation) of **John** is identified with a class of domain subsets. (Talk of intensions, extensions, satisfaction, and membership is relative to a given intensional model of the intensional language in question.) Extensions as in (2) are of course none other than type-$\langle 1\rangle$ quantifiers in our sense. Regarding (2), Keenan and

[2]Zermelo gives a different notion of rank at the end of the second paragraph of [14]: $\mathrm{rank}(\mathcal{A})$ is defined there as the unique α with $\mathcal{A} \in V_{\alpha+1} \setminus V_\alpha$. So $\bigvee_{n<\omega} \neg^n(Ra)$ would be of rank ω according to this definition. This definition is preferable to our own to the extent that, according to it, every ordinal serves as rank.

Westerståhl cite Montague's [7] (see their [5]). However, we do not see that this notion is playing any role in [7]. Rather, the actual source seems to be Basic Translation Rule T1(d) in [8].

We justify a remark following Definition 5.1 of SPLQ by formulating and proving

THEOREM 5.1 (see [5] Generalization 1.1.2(i)). *Let domain $\mathfrak{D}$ be given. Then each member of $\mathcal{Q}_{\mathfrak{D}}\langle\vec{n}\rangle$ is a Boolean combination of members of $\mathcal{M}_{\mathfrak{D}}\langle\vec{n}\rangle$.*

PROOF. It is sufficient to see that each atom $Q^{A_1,\dots,A_p}$ of $\mathcal{Q}_{\mathfrak{D}}\langle\vec{n}\rangle$ can be expressed as meet

$$\bigcurlywedge_{\langle\mathfrak{a}_1,\dots,\mathfrak{a}_{n_1}\rangle\in A_1} H_1^{\mathfrak{a}_1,\dots,\mathfrak{a}_{n_1}} \curlywedge \bigcurlywedge_{\langle\mathfrak{a}_1,\dots,\mathfrak{a}_{n_1}\rangle\notin A_1} {\sim}H_1^{\mathfrak{a}_1,\dots,\mathfrak{a}_{n_1}} \curlywedge \dots \curlywedge$$
$$\bigcurlywedge_{\langle\mathfrak{a}_1,\dots,\mathfrak{a}_{n_p}\rangle\in A_p} H_p^{\mathfrak{a}_1,\dots,\mathfrak{a}_{n_p}} \curlywedge \bigcurlywedge_{\langle\mathfrak{a}_1,\dots,\mathfrak{a}_{n_p}\rangle\notin A_p} {\sim}H_p^{\mathfrak{a}_1,\dots,\mathfrak{a}_{n_p}}.$$

⊣

It follows immediately from Theorem 5.1 and θ-closure that if $|\mathfrak{D}| < \theta$ then each unitary type-$\langle\vec{n}\rangle$ over $\mathfrak{D}$ is a member of $\mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$. (Note that the meet presented in the proof of Theorem 5.1 involves fewer than θ conjuncts in that case.) We have shown that $\left\langle \mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle, \bigcurlyvee, \bigcurlywedge, \sim, \mathsf{false}^{\vec{n}}, \mathsf{true}^{\vec{n}} \right\rangle$ is atomic if $|\mathfrak{D}| < \theta$, thereby justifying the claim following Definition 5.2 of SPLQ.

If, instead, $|\mathfrak{D}| \geq \theta$, then we argue as follows. As a preliminary, we define the set $\mathrm{Constit}(Q^{\vec{n}}) \subseteq \mathfrak{D}$ of *constituents of type-$\langle\vec{n}\rangle$ quantifier $Q^{\vec{n}}$ over $\mathfrak{D}$* on analogy with Definition 4.10.

DEFINITION 5.2. *Let domain $\mathfrak{D}$ and signature $\Sigma = \langle R_1, \dots, R_p\rangle$ be given. Let $Q^{\vec{n}} \in \mathcal{Q}_{\mathfrak{D}}\langle\vec{n}\rangle$. Then the set* $\mathrm{Constit}(Q^{\vec{n}})$ *of* constituents of $Q^{\vec{n}}$ *is defined as follows.*

1. *If $Q^{\vec{n}}$ is montagovian n_ℓ-tuple $H_\ell^{\mathfrak{a}_1,\dots,\mathfrak{a}_{n_\ell}}$ with $1 \leq \ell \leq p$, then* $\mathrm{Constit}(Q^{\vec{n}}) = \{\mathfrak{a}_1, \dots, \mathfrak{a}_{n_\ell}\}$.
2. *If $Q^{\vec{n}}$ is ${\sim}\tilde{Q}^{\vec{n}}$ with $\tilde{Q}^{\vec{n}} \in \mathcal{Q}_{\mathfrak{D}}\langle\vec{n}\rangle$, then* $\mathrm{Constit}(Q^{\vec{n}}) = \mathrm{Constit}(\tilde{Q}^{\vec{n}})$.
3. *If $Q^{\vec{n}}$ is $\bigcurlyvee \mathcal{K}$ with $\mathcal{K} \subseteq \mathcal{Q}_{\mathfrak{D}}\langle\vec{n}\rangle$, then* $\mathrm{Constit}(Q^{\vec{n}}) = \bigcup_{\tilde{Q}^{\vec{n}}\in\mathcal{K}} \mathrm{Constit}(\tilde{Q}^{\vec{n}})$.
4. *If $Q^{\vec{n}}$ is $\bigcurlywedge \mathcal{K}$ with $\mathcal{K} \subseteq \mathcal{Q}_{\mathfrak{D}}\langle\vec{n}\rangle$, then* $\mathrm{Constit}(Q^{\vec{n}}) = \bigcup_{\tilde{Q}^{\vec{n}}\in\mathcal{K}} \mathrm{Constit}(\tilde{Q}^{\vec{n}})$.

Next, by an easy argument by induction on the complexity of $Q^{\vec{n}} \in \mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ whereby we exploit the fact that the members of $\mathcal{M}_{\mathfrak{D}}\langle\vec{n}\rangle$ are generators of $\mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$, we can show that the constituent set of any $Q^{\vec{n}} \in \mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ satisfies $|\mathrm{Constit}(Q^{\vec{n}})| < \theta$. Suppose for the sake of contradiction that $\mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ contains an atom, that is, a minimal member $Q^{\vec{n}}$ of $\mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle \setminus \{\mathsf{false}^{\vec{n}}\}$. This means that $Q^{\vec{n}}$ must satisfy $Q^{\vec{n}} \neq \mathsf{false}^{\vec{n}}$ and that $\tilde{Q}^{\vec{n}} \preceq Q^{\vec{n}}$ implies either $\tilde{Q}^{\vec{n}}$ unsatisfiable or $\tilde{Q}^{\vec{n}} = Q^{\vec{n}}$. Since $|\mathfrak{D}| \geq \theta$ by assumption, there exists $\mathfrak{a} \in \mathfrak{D} \setminus \mathrm{Constit}(Q^{\vec{n}}) \neq \emptyset$. By the analogue of Theorem 4.13 it can be seen that $Q^{\vec{n}} \curlywedge H_\ell^{\mathfrak{a},\dots,\mathfrak{a}}$ is satisfiable with $Q^{\vec{n}} \curlywedge H_\ell^{\mathfrak{a},\dots,\mathfrak{a}} \neq Q^{\vec{n}}$, where $1 \leq \ell \leq p$ and $H_\ell^{\mathfrak{a},\dots,\mathfrak{a}} \in \mathcal{M}_{\mathfrak{D}}\langle\vec{n}\rangle \subseteq \mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$. It follows that $\mathsf{false}^{\vec{n}} \prec Q^{\vec{n}} \curlywedge H_\ell^{\mathfrak{a},\dots,\mathfrak{a}} \prec Q^{\vec{n}}$, which contradicts $Q^{\vec{n}}$ an atom.

We justify a remark following Definition 5.2 of SPLQ to the effect that the mapping $\varrho : \mathcal{H}_{\mathfrak{D},\Sigma} \mapsto \mathcal{Q}_{\mathfrak{D}}\langle\vec{n}\rangle$ defined by $\varrho(\mathcal{A}) = Q^{\vec{n}}_{\mathcal{A}}$ is a homomorphism.

1. We have that $\rho(\neg\mathcal{A}) = Q^{\vec{n}}_{\neg\mathcal{A}} = {\sim}Q^{\vec{n}}_{\mathcal{A}} = {\sim}\rho(\mathcal{A})$.
2. We have that $\rho(\bigvee K) = Q^{\vec{n}}_{\bigvee K} = \curlyvee\{Q^{\vec{n}}_{\mathcal{B}} \mid \mathcal{B} \in K\} = \curlyvee\{\rho(\mathcal{B}) \mid \mathcal{B} \in K\} = \curlyvee \rho(K)$.
3. We have that $\rho(\bigwedge K) = Q^{\vec{n}}_{\bigwedge K} = \curlywedge\{Q^{\vec{n}}_{\mathcal{B}} \mid \mathcal{B} \in K\} = \curlywedge\{\rho(\mathcal{B}) \mid \mathcal{B} \in K\} = \curlywedge \rho(K)$.

Next, we show that $\rho \restriction \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ is surjective onto $\mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$. Suppose that $Q^{\vec{n}} \in \mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$.

1. If $Q^{\vec{n}}$ is montagovian n_ℓ-tuple $H_\ell^{\mathfrak{a}_1,\dots,\mathfrak{a}_{n_\ell}}$ with $1 \leq \ell \leq p$, then $Q^{\vec{n}}$ is $\rho(R_\ell\mathfrak{a}_1\dots\mathfrak{a}_{n_\ell})$ with $R_\ell\mathfrak{a}_1\dots\mathfrak{a}_{n_\ell}$ in $\mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$.
2. If $Q^{\vec{n}}$ is ${\sim}\tilde{Q}^{\vec{n}}$ with $\tilde{Q}^{\vec{n}} \in \mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$, then $Q^{\vec{n}}$ is $\rho(\neg\rho^{-1}(\tilde{Q}^{\vec{n}}))$, where $\rho^{-1}(\tilde{Q}^{\vec{n}})$, and hence $\neg\rho^{-1}(\tilde{Q}^{\vec{n}})$, is defined and a member of $\mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ by induction hypothesis.
3. If $Q^{\vec{n}}$ is $\curlyvee \mathcal{K}$ with $\mathcal{K} \subseteq \mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ and $|\mathcal{K}| < \theta$, then $Q^{\vec{n}}$ is $\rho\big(\bigvee \rho^{-1}(\mathcal{K})\big)$, where each member of $\rho^{-1}(\mathcal{K})$, and hence $\bigvee \rho^{-1}(\mathcal{K})$, is defined and a member of $\mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ by induction hypothesis and Theorem 2.4.
4. If $Q^{\vec{n}}$ is $\curlywedge \mathcal{K}$ with $\mathcal{K} \subseteq \mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ and $|\mathcal{K}| < \theta$, then $Q^{\vec{n}}$ is $\rho\big(\bigwedge \rho^{-1}(\mathcal{K})\big)$, where each member of $\rho^{-1}(\mathcal{K})$, and hence $\bigwedge \rho^{-1}(\mathcal{K})$, is defined and a member of $\mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ by induction hypothesis and Theorem 2.4.

As a direct consequence of the definition of $\preceq$, we have

THEOREM 5.3. *Let domain $\mathfrak{D}$ be given and let $Q^{\vec{n}}, \tilde{Q}^{\vec{n}} \in \mathcal{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$. Then $Q^{\vec{n}} \preceq \tilde{Q}^{\vec{n}}$ if and only if* $Q^{\vec{n}}(A_1,\dots,A_p) = \text{true}$ *implies* $\tilde{Q}^{\vec{n}}(A_1,\dots,A_p) = \text{true}$ *for all $A_1 \in \wp(\mathfrak{D}^{n_1})$, ... , and all $A_p \in \wp(\mathfrak{D}^{n_p})$.*

THEOREM 5.4 (Theorem 5.3 of SPLQ). *Suppose that we are given domain $\mathfrak{D}$ and signature $\Sigma = \langle R_1,\dots,R_p\rangle$, and let proposition $\mathcal{A} \in \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$. Then we have*

$$\tau \in \operatorname{Mod}(\mathcal{A}) \textit{ if and only if } Q^{\vec{n}}_{\mathcal{A}}(f_1^{-1}(1),\dots,f_p^{-1}(1)) = \text{true}$$

for any $\tau =: \langle f_1,\dots,f_p\rangle \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$.

PROOF. Our proof proceeds by transfinite induction on the complexity of proposition $\mathcal{A}$. As base case, suppose $\mathcal{A}$ is rank-0 proposition $R_\ell\mathfrak{a}_1\dots\mathfrak{a}_{n_\ell}$. Then $Q^{\vec{n}}_{\mathcal{A}}$ is as in clause (1) of Definition 5.2.1 of SPLQ. We have

$$\begin{aligned} & Q^{\vec{n}}_{R_\ell\mathfrak{a}_1\dots\mathfrak{a}_{n_\ell}}(f_1^{-1}(1),\dots,f_p^{-1}(1)) = \text{true} \\ \Longleftrightarrow\ & \langle\mathfrak{a}_1,\dots,\mathfrak{a}_{n_\ell}\rangle \in f_\ell^{-1}(1) \\ \Longleftrightarrow\ & f_\ell(\mathfrak{a}_1,\dots,\mathfrak{a}_{n_\ell}) = 1 \\ \Longleftrightarrow\ & \operatorname{val}(R_\ell\mathfrak{a}_1\dots\mathfrak{a}_{n_\ell},\tau) = \text{true} \end{aligned}$$

by Definitions 3.1 and 3.2 of SPLQ

$$\begin{aligned} \Longleftrightarrow\ & \tau \models R_\ell\mathfrak{a}_1\dots\mathfrak{a}_{n_\ell} \\ \Longleftrightarrow\ & \tau \in \operatorname{Mod}(R_\ell\mathfrak{a}_1\dots\mathfrak{a}_{n_\ell}). \end{aligned}$$

Two inductive subcases are straightforward. First, suppose that $\mathcal{A}$ is of the form $\neg\mathcal{B}$. Then

$$\begin{aligned} & Q^{\vec{n}}_{\mathcal{A}}(f_1^{-1}(1), \dots, f_p^{-1}(1)) = \text{true} \\ \Longleftrightarrow\ & {\sim}Q^{\vec{n}}_{\mathcal{B}}(f_1^{-1}(1), \dots, f_p^{-1}(1)) = \text{true} \end{aligned}$$

by Definition 5.2 of SPLQ

$$\Longleftrightarrow Q^{\vec{n}}_{\mathcal{B}}(f_1^{-1}(1), \dots, f_p^{-1}(1)) = \text{false}$$

by $(3_{\mathfrak{D}})$

$$\Longleftrightarrow \tau \notin \text{Mod}(\mathcal{B})$$

by induction hypothesis

$$\Longleftrightarrow \tau \in \text{Mod}(\mathcal{A})$$

by Theorem 3.1.1.

Next, suppose that $\mathcal{A}$ is of the form $\bigvee K$ with $K \subseteq \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ and $|K| < \theta$. Then

$$\begin{aligned} & Q^{\vec{n}}_{\mathcal{A}}(f_1^{-1}(1), \dots, f_p^{-1}(1)) = \text{true} \\ \Longleftrightarrow\ & \Big[\bigvee\{Q^{\vec{n}}_{\mathcal{B}} \mid \mathcal{B} \in K\}\Big](f_1^{-1}(1), \dots, f_p^{-1}(1)) = \text{true} \end{aligned}$$

by Definition 5.2 SPLQ

$$\Longleftrightarrow Q^{\vec{n}}_{\mathcal{B}^*}(f_1^{-1}(1), \dots, f_p^{-1}(1)) = \text{true for some } \mathcal{B}^* \in K$$

by $(1_{\mathfrak{D}})$

$$\Longleftrightarrow \tau \in \text{Mod}(\mathcal{B}^*) \text{ for some } \mathcal{B}^* \in K$$

by induction hypothesis

$$\Longleftrightarrow \tau \in \text{Mod}(\mathcal{A})$$

by Theorem 3.1.2.

$\dashv$

Theorem 5.5 (Theorem 5.4 of SPLQ). *Suppose that we are given domain $\mathfrak{D}$ and signature $\Sigma = \langle R_1, \dots, R_p\rangle$. Let $\tau = \langle f_1, \dots, f_p\rangle \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$. Where $A_\ell \in \wp(\mathfrak{D}^\ell)$ for $1 \le \ell \le p$, we have $Q^{\vec{n}}_{\text{eldg}(\tau)}(A_1, \dots, A_p) = \text{true}$ if and only if $A_\ell = f_\ell^{-1}(1)$ for all $1 \le \ell \le p$.*

PROOF.

$$Q^{\vec{n}}_{\mathrm{eldg}(\tau)}(A_1,\ldots,A_p)=\mathrm{true}\Longleftrightarrow\langle\chi_{A_1},\ldots,\chi_{A_p}\rangle\in\mathrm{Mod}(\mathrm{eldg}(\tau))$$

by Theorem 5.4 since characteristic function χ_{A_ℓ} satisfies $\chi^{-1}_{A_\ell}(1)=A_\ell$ for $1\le\ell\le p$

$$\Longleftrightarrow\langle\chi_{A_1},\ldots,\chi_{A_p}\rangle=\tau$$

since $\mathrm{Mod}(\mathrm{eldg}(\tau))=\{\tau\}$

$$\Longleftrightarrow\chi_{A_\ell}=f_\ell\text{ for all }1\le\ell\le p$$
$$\Longleftrightarrow A_\ell=f^{-1}_\ell(1)\text{ for all }1\le\ell\le p$$

since $\chi^{-1}_{A_\ell}(1)=A_\ell$ for all $1\le\ell\le p$.

⊣

THEOREM 5.6. *Suppose that domain* $\mathfrak{D}$ *and signature* $\Sigma=\langle R_1,\ldots,R_p\rangle$ *are given. Let propositions* $\mathcal{A},\mathcal{B}\in\mathcal{H}^\theta_{\mathfrak{D},\Sigma}$. *Then* $\mathcal{A}\Rightarrow\mathcal{B}$ *if and only if* $Q^{\vec{n}}_{\mathcal{A}}\preceq Q^{\vec{n}}_{\mathcal{B}}$.

PROOF. We can write

$$\mathcal{A}\Rightarrow\mathcal{B}$$
$$\Longleftrightarrow\mathrm{Mod}(\mathcal{A})\subseteq\mathrm{Mod}(\mathcal{B})$$
$$\Longleftrightarrow\text{given }\tau\in\mathfrak{T}^{\vec{n}}_{\mathfrak{D}}\text{ arbitrary},\tau\in\mathrm{Mod}(\mathcal{A})\text{ implies }\tau\in\mathrm{Mod}(\mathcal{B})$$
$$\Longleftrightarrow\text{given }\tau=\langle f_1,\ldots,f_p\rangle\in\mathfrak{T}^{\vec{n}}_{\mathfrak{D}}\text{ arbitrary},$$
$$Q^{\vec{n}}_{\mathcal{A}}(f^{-1}_1(1),\ldots,f^{-1}_p(1))=\mathrm{true}\text{ implies }Q^{\vec{n}}_{\mathcal{B}}(f^{-1}_1(1),\ldots,f^{-1}_p(1))=\mathrm{true}$$

by Theorem 5.4

$$\Longleftrightarrow\text{given }A_1\in\wp(\mathfrak{D}^{n_1}),\ldots,A_p\in\wp(\mathfrak{D}^{n_p})\text{ arbitrary},$$
$$Q^{\vec{n}}_{\mathcal{A}}(A_1,\ldots,A_p)=\mathrm{true}\text{ implies }Q^{\vec{n}}_{\mathcal{B}}(A_1,\ldots,A_p)=\mathrm{true}$$
$$\Longleftrightarrow\text{given }A_1\in\wp(\mathfrak{D}^{n_1}),\ldots,A_p\in\wp(\mathfrak{D}^{n_p})\text{ arbitrary},$$
$$\text{either }Q^{\vec{n}}_{\mathcal{A}}(A_1,\ldots,A_p)=\mathrm{true}\text{ or }Q^{\vec{n}}_{\mathcal{B}}(A_1,\ldots,A_p)=\mathrm{true}$$
$$\text{if and only if }Q^{\vec{n}}_{\mathcal{B}}(A_1,\ldots,A_p)=\mathrm{true}$$
$$\Longleftrightarrow\text{given }A_1\in\wp(\mathfrak{D}^{n_1}),\ldots,A_p\in\wp(\mathfrak{D}^{n_p})\text{ arbitrary},$$
$$[Q^{\vec{n}}_{\mathcal{A}}\curlyvee Q^{\vec{n}}_{\mathcal{B}}](A_1,\ldots,A_p)=\mathrm{true}\text{ if and only if }Q^{\vec{n}}_{\mathcal{B}}(A_1,\ldots,A_p)=\mathrm{true}$$

by $(1_{\mathfrak{D}})$

$$\Longleftrightarrow Q^{\vec{n}}_{\mathcal{A}}\curlyvee Q^{\vec{n}}_{\mathcal{B}}=Q^{\vec{n}}_{\mathcal{B}}$$
$$\Longleftrightarrow Q^{\vec{n}}_{\mathcal{A}}\preceq Q^{\vec{n}}_{\mathcal{B}}.$$

⊣

A remark at the end of §5 of SPLQ is justified by

COROLLARY 5.7. *Suppose that domain $\mathfrak{D}$ and signature $\Sigma = \langle R_1, \dots, R_p\rangle$ are given. Let propositions $\mathcal{A}, \mathcal{B} \in \mathcal{H}^\theta_{\mathfrak{D},\Sigma}$. Then $\mathcal{A} \Leftrightarrow \mathcal{B}$ if and only if $Q^{\vec{n}}_{\mathcal{A}} = Q^{\vec{n}}_{\mathcal{B}}$.*

THEOREM 5.8 (Theorem 5.5 of SPLQ). *Suppose that domain $\mathfrak{D}$ and signature $\Sigma = \langle R_1, \dots, R_p\rangle$ are given. Let $\tau \in \mathfrak{T}^{\vec{n}}_{\mathfrak{D}}$ and let $\pi \in S_{\mathfrak{D}}$. Then with $A_\ell \in \wp(\mathfrak{D}^\ell)$ for all $1 \le \ell \le p$ we have*

$$Q^{\vec{n}}_{\mathrm{eldg}(\tau)}(A_1, \dots, A_p) = Q^{\vec{n}}_{\mathrm{eldg}(\pi(\tau))}(\pi(A_1), \dots, \pi(A_p)).$$

PROOF. It is sufficient to show that $[Q^{\vec{n}}_{\mathrm{eldg}(\tau)}](A_1, \dots, A_p) = \text{true}$ implies $[Q^{\vec{n}}_{\mathrm{eldg}(\pi(\tau))}](\pi(A_1), \dots, \pi(A_p)) = \text{true}$. Suppose the former. Let $\tau = \langle f_1, \dots, f_p\rangle$. Theorem 5.5 implies that $A_\ell = f_\ell^{-1}(1)$ for $1 \le \ell \le p$. But then $\pi(A_\ell) = \pi(f_\ell^{-1}(1)) = \pi(f_\ell)^{-1}(1)$ so that $[Q^{\vec{n}}_{\mathrm{eldg}(\pi(\tau))}](\pi(A_1), \dots, \pi(A_p)) = \text{true}$ by Theorem 5.5 again. ⊣

The following example illustrates Theorem 5.8.

EXAMPLE 5.9. Let $\mathfrak{D} = \{a, b\}$ and let $\Sigma = \langle R\rangle$ with R monadic. Let $\tau = \langle f\rangle \in \mathfrak{T}^1_{\{a,b\}}$, where $f(a) = 1$ and $f(b) = 0$. Thus $\mathrm{eldg}(\tau)$ is $Ra \wedge \neg Rb$ so that quantifier image $Q^1_{\mathrm{eldg}(\tau)}$ is $H^a_1 \curlywedge \sim H^b_1$. Suppose $Q^1_{\mathrm{eldg}(\tau)}(A) = \text{true}$ with $A \in \wp(\mathfrak{D})$. This must mean that $A = \{a\}$. Let $\pi = (a\ b)$ so that $\pi(\tau) = \langle \pi(f)\rangle$ and $\pi(A) = \{b\}$. Then $\mathrm{eldg}(\pi(\tau))$ is $Rb \wedge \neg Ra$ so that $Q^1_{\mathrm{eldg}(\pi(\tau))}$ is $H^b_1 \curlywedge \sim H^a_1$. What of $Q^1_{\mathrm{eldg}(\pi(\tau))}(\pi(A)) = H^b_1 \curlywedge \sim H^a_1(\{b\})$? Clearly, it has value true, as predicted by Theorem 5.8.

§6. Logical and categorical quantifiers. That both $\mathsf{false}^{1,1}$ and $\mathsf{true}^{1,1}$ are logical quantifiers should be clear.

THEOREM 6.1 (First Representation Theorem 6.5 of SPLQ). *Let domain $\mathfrak{D}$ be given and let signature $\Sigma = \langle R_1, \dots, R_p\rangle$. Suppose $\mathcal{A} \in \mathcal{H}^\theta_{\mathfrak{D},\Sigma}$. Then proposition $\mathcal{A} \in \mathcal{H}^\theta_{\mathfrak{D},\Sigma}$ is symmetric over $\mathfrak{D}$ and Σ if and only if quantifier image $Q^{\vec{n}}_{\mathcal{A}}$ is a logical type-$\langle\vec{n}\rangle$ quantifier over $\mathfrak{D}$ and Σ.*

PROOF. Suppose proposition $\mathcal{A}$ is symmetric. If $\mathcal{A}$ is a tautology (contradiction), then $Q^{\vec{n}}_{\mathcal{A}}$ is logical quantifier $\mathsf{true}^{\vec{n}}$ ($\mathsf{false}^{\vec{n}}$). So suppose that $\mathcal{A}$ is contingent, from which it follows that θ exceeds $|\mathfrak{D}|$ and hence $|\mathrm{Mod}(\mathcal{A})|$. By Corollary 5.7 we may assume $\mathcal{A}$ to be $\bigvee_{\tau\in\mathrm{Mod}(\mathcal{A})} \mathrm{eldg}(\tau) \in \mathcal{H}^\theta_{\mathfrak{D},\Sigma}$. In that case, by Definition 5.2 of SPLQ, $Q^{\vec{n}}_{\mathcal{A}}$ is $\curlyvee_{\tau\in\mathrm{Mod}(\mathcal{A})} Q^{\vec{n}}_{\mathrm{eldg}(\tau)}$. Let $Q^{\vec{n}}_{\mathcal{A}}(B_1, \dots, B_p) = \text{true}$ with $B_1 \in \wp(\mathfrak{D}^{n_1}), \dots, B_p \in \wp(\mathfrak{D}^{n_p})$. We have that $Q^{\vec{n}}_{\mathrm{eldg}(\tau^*)}(B_1, \dots, B_p) = \text{true}$ for some $\tau^* \in \mathrm{Mod}(\mathcal{A})$. With $\pi \in S_{\mathfrak{D}}$ arbitrary, Theorem 5.8 then gives $Q^{\vec{n}}_{\mathrm{eldg}(\pi(\tau^*))}(\pi(B_1), \dots, \pi(B_p)) = \text{true}$ with $\pi(\tau^*) \in \mathrm{Mod}(\mathcal{A})$ by symmetry. So $Q^{\vec{n}}_{\mathcal{A}}(\pi(B_1), \dots, \pi(B_p)) = \text{true}$, which is sufficient to see that $Q^{\vec{n}}_{\mathcal{A}}$ is logical.

Suppose $Q^{\vec{n}}_{\mathcal{A}}$ is a logical quantifier. If $Q^{\vec{n}}_{\mathcal{A}}$ is $\mathsf{true}^{\vec{n}}_{\mathfrak{D}}$ ($\mathsf{false}^{\vec{n}}_{\mathfrak{D}}$), then $\mathcal{A}$ is a tautology (contradiction) and hence symmetric. Otherwise, we have $|\mathfrak{D}| < \theta$ and can again assume that $\mathcal{A}$ is $\bigvee_{\tau\in\mathrm{Mod}(\mathcal{A})} \mathrm{eldg}(\tau)$ so that $Q^{\vec{n}}_{\mathcal{A}}$ is $\curlyvee_{\tau\in\mathrm{Mod}(\mathcal{A})} Q^{\vec{n}}_{\mathrm{eldg}(\tau)}$. Suppose that $\tau^* \in \mathrm{Mod}(\mathcal{A})$. By Theorem 3.2 we have $\mathrm{eldg}(\tau^*) \Rightarrow \mathcal{A}$. Then by Theorem 5.6 we have $Q^{\vec{n}}_{\mathrm{eldg}(\tau^*)} \preceq Q^{\vec{n}}_{\mathcal{A}}$. Let $\pi \in S_{\mathfrak{D}}$. It follows that, for

$B_1 \in \wp(\mathfrak{D}^{n_1}), \ldots, B_p \in \wp(\mathfrak{D}^{n_p})$, we have

$$Q^{\vec{n}}_{\mathrm{eldg}(\pi(\tau^*))}(B_1, \ldots, B_p) = \text{true}$$
$$\Longrightarrow Q^{\vec{n}}_{\mathrm{eldg}(\tau^*)}(\pi^{-1}(B_1), \ldots, \pi^{-1}(B_p)) = \text{true}$$

by Theorem 5.8

$$\Longrightarrow Q^{\vec{n}}_{\mathcal{A}}(\pi^{-1}(B_1), \ldots, \pi^{-1}(B_p)) = \text{true}$$

by Theorem 5.3 since $Q^{\vec{n}}_{\mathrm{eldg}(\tau^*)} \preceq Q^{\vec{n}}_{\mathcal{A}}$

$$\Longrightarrow Q^{\vec{n}}_{\mathcal{A}}(B_1, \ldots, B_p) = \text{true}$$

since $Q^{\vec{n}}_{\mathcal{A}}$ is logical.

Now by Theorem 5.3 we have that $Q^{\vec{n}}_{\mathrm{eldg}(\pi(\tau^*))} \preceq Q^{\vec{n}}_{\mathcal{A}}$, and it follows, by Theorem 5.6 again, that $\mathrm{eldg}(\pi(\tau^*)) \Rightarrow \mathcal{A}$. But then by Theorem 3.2 we have $\pi(\tau^*) \in \mathrm{Mod}(\mathcal{A})$. Since π was arbitrary, $\mathrm{Mod}(\mathcal{A})$ is seen to be permutation-invariant. ⊣

We give logic versions of our representation theorems. They have system versions as consequences, assuming an unbounded sequence of strongly inaccessible cardinals. In the case of First Representation Theorem 6.1 we have

THEOREM 6.2 (First Representation Theorem (system version)). *Let domain $\mathfrak{D}$ be given and let signature $\Sigma = \langle R_1, \ldots, R_p\rangle$. Then proposition $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},\Sigma}$ is symmetric over $\mathfrak{D}$ and Σ if and only if quantifier image $Q^{\vec{n}}_{\mathcal{A}}$ is a logical type-$\langle\vec{n}\rangle$ quantifier over $\mathfrak{D}$ and Σ.*

PROOF. Let $\mathcal{A} \in \mathcal{H}_{\mathfrak{D},\Sigma}$. Then there exists strongly inaccessible θ with $\mathcal{A} \in \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$ so that $Q^{\vec{n}}_{\mathcal{A}} \in \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ by a remark following Definition 5.2 of SPLQ. Then we can write

$$\mathcal{A} \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$$
$$\Longleftrightarrow \mathcal{A} \in \mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma}$$

since $\mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma} =: \mathrm{Sym}_{\mathfrak{D},\Sigma} \cap \mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$

$$\Longleftrightarrow Q^{\vec{n}}_{\mathcal{A}} \in \mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$$

by Representation Theorem 6.1

$$\Longleftrightarrow Q^{\vec{n}}_{\mathcal{A}} \in \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$$

since $Q^{\vec{n}}_{\mathcal{A}} \in \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ and $\mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle =: \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle \cap \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$.

⊣

Note that $|\mathfrak{D}| < \theta$ implies $\mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle = \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ and $\mathrm{CQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle = \mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle$.

It is obvious that $Q^{[A_1,\dots,A_p]}$ and $Q^{[\pi(A_1),\dots,\pi(A_p)]}$ are identical quasi-unitary quantifiers for any $\pi \in S_{\mathfrak{D}}$.

THEOREM 6.3. *Any categorical type-$\langle\vec{n}\rangle$ quantifier $Q^{\vec{n}}$ is a quasi-unitary type-$\langle\vec{n}\rangle$ quantifier, and vice versa.*

PROOF. For simplicity we assume $p = 2$. Suppose that $Q^{\vec{n}}$ is not quasi-unitary. If $Q^{\vec{n}}$ is unsatisfiable, then it is not categorical, and we are done. So assume that $Q^{\vec{n}}$ is satisfiable. It follows that $Q^{\vec{n}}(A, B) =$ true for some $A \in \wp(\mathfrak{D}^{n_1})$ and some $B \in \wp(\mathfrak{D}^{n_2})$. But, in this context, $Q^{\vec{n}}$ not quasi-unitary means one of two things. The first possibility is that $Q^{\vec{n}}(\pi(A), \pi(B)) =$ false for some $\pi \in S_{\mathfrak{D}}$, in which case $Q^{\vec{n}}$ is not permutation-invariant and hence not categorical. The second possibility is that, although permutation-invariant, $Q^{\vec{n}}$ maps some pair $\langle A', B'\rangle$ to true despite its not being the case that $\pi(A) = A'$ and $\pi(B) = B'$ for any $\pi \in S_{\mathfrak{D}}$. In this second case, $Q^{\vec{n}}$ is not "minimal" in the relevant sense and hence not categorical: there exists logical $\tilde{Q}^{\vec{n}}$ satisfiable with $\tilde{Q}^{\vec{n}} < Q^{\vec{n}}$.

For the other direction, suppose that $Q^{\vec{n}}$ is quasi-unitary $Q^{A,B}$ for fixed $A \in \wp(\mathfrak{D}^{n_1})$ and $B \in \wp(\mathfrak{D}^{n_2})$. Assume, for the sake of contradiction, that satisfiable $\tilde{Q}^{\vec{n}}$ is logical with $\tilde{Q}^{\vec{n}} \prec Q^{\vec{n}}$. It follows that there exist $A' \in \wp(\mathfrak{D}^{n_1})$ and $B' \in \wp(D^{n_2})$ such that $\tilde{Q}^{\vec{n}}(A', B') =$ false while $Q^{\vec{n}}(A', B') =$ true so that $\pi(A) = A'$ and $\pi(B) = B'$ for some $\pi \in S_{\mathfrak{D}}$. Next, let A^* and B^* be arbitrary elements of $\wp(\mathfrak{D}^{n_1})$ and $\wp(\mathfrak{D}^{n_2})$, respectively. Now either $Q^{\vec{n}}(A^*, B^*) =$ true or $Q^{\vec{n}}(A^*, B^*) =$ false. If the latter, then $\tilde{Q}^{\vec{n}}(A^*, B^*) =$ false also since $\tilde{Q}^{\vec{n}} \prec Q^{\vec{n}}$. If the former, then, for some $\pi^* \in S_{\mathfrak{D}}$, we have $\pi^*(A) = A^*$ and $\pi^*(B) = B^*$. It follows that $A^* = \pi^*(\pi^{-1}(A'))$ and $B^* = \pi^*(\pi^{-1}(B'))$. So $\tilde{Q}^{\vec{n}}(A', B') =$ false together with invariance implies that $\tilde{Q}^{\vec{n}}(A^*, B^*) =$ false. But A^* and B^* were arbitrary, which contradicts $\tilde{Q}^{\vec{n}}$ satisfiable. ⊣

THEOREM 6.4. *Where $|\mathfrak{D}| = 1$, quantifier $Q^{\vec{n}}$ is a categorical quantifier over $\mathfrak{D}$ just in case it is a unitary quantifier over $\mathfrak{D}$.*

PROOF. If $Q^{\vec{n}}$ is unitary, then it is satisfiable and $\tilde{Q}^{\vec{n}} \prec Q^{\vec{n}}$ implies $\tilde{Q}^{\vec{n}} = \mathsf{false}^{\vec{n}}$. For the other direction, by assumption, there are no nontrivial domain permutations. Consequently, any quantifier whatever is permutation-invariant and hence logical so that any $\tilde{Q}^{\vec{n}} \prec Q^{\vec{n}}$ with $Q^{\vec{n}}$ categorical must be unsatisfiable, which means that $Q^{\vec{n}}$ is unitary in fact. ⊣

We justify the remark following Definition 6.6 of SPLQ.

THEOREM 6.5. *Quantifier $Q^{\vec{n}} \in \mathfrak{Q}_{\mathfrak{D}}\langle\vec{n}\rangle \in \mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ if and only if $Q^{\vec{n}}$ is satisfiable and $Q^{\vec{n}} \in \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ and any satisfiable $\tilde{Q}^{\vec{n}} \in \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ with $\tilde{Q}^{\vec{n}} \preceq Q^{\vec{n}}$ is such that $\tilde{Q}^{\vec{n}} = Q^{\vec{n}}$ in fact.*

PROOF. Let $Q^{\vec{n}} \in \mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle$. By Definition 6.6 of SPLQ we have that $Q^{\vec{n}}$ is both satisfiable and logical. Suppose, for the sake of contradiction, that there exists satisfiable logical $\tilde{Q}^{\vec{n}}$ with $\tilde{Q}^{\vec{n}} \prec Q^{\vec{n}}$. By Definition 6.6 again we have that $\tilde{Q}^{\vec{n}}$ is nonlogical, which is a contradiction.

For the other direction, suppose that $Q^{\vec{n}}$ is satisfiable and logical but not in $\mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle$. It follows that there exists satisfiable logical $\tilde{Q}^{\vec{n}} \prec Q^{\vec{n}}$, and we are done. ⊣

The logic version of Theorem 6.5 splits into two cases.

THEOREM 6.6. *Let* $|\mathfrak{D}| < \theta$. *Then* $Q^{\vec{n}} \in \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ *is in* $\mathrm{CQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ *if and only if* $Q^{\vec{n}}$ *is satisfiable and* $Q^{\vec{n}} \in \mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ *and any satisfiable* $\tilde{Q}^{\vec{n}} \in \mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ *with* $\tilde{Q}^{\vec{n}} \preceq Q^{\vec{n}}$ *is such that* $\tilde{Q}^{\vec{n}} = Q^{\vec{n}}$ *in fact.*

PROOF. Let $Q^{\vec{n}} \in \mathrm{CQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle = \mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle \cap \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle \subseteq \mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle$. By Theorem 6.5 we have that $Q^{\vec{n}}$ is satisfiable and $Q^{\vec{n}} \in \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$. So $Q^{\vec{n}} \in \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle \cap \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle = \mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$. Suppose that $\tilde{Q}^{\vec{n}} \in \mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ is satisfiable with $\tilde{Q}^{\vec{n}} \preceq Q^{\vec{n}}$. Then, by Theorem 6.5 again, $\tilde{Q}^{\vec{n}} \in \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ so that $\tilde{Q}^{\vec{n}} = Q^{\vec{n}}$.

For the other direction, note first that $|\mathfrak{D}| < \theta$ implies that (1) $|\wp(\mathfrak{D}^{n_1}) \times \ldots \times \wp(\mathfrak{D}^{n_p})| < \theta$ and (2) $\sum_{\ell=1}^{p} |\mathfrak{D}^{n_\ell}| < \theta$. Suppose that $Q^{\vec{n}} \in \mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle \setminus \mathrm{CQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ is satisfiable. Then $Q^{\vec{n}} \notin \mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ since $Q^{\vec{n}} \in \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ by assumption. (Recall that $\mathrm{CQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle =: \mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle \cap \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$.) So there exists satisfiable $\tilde{Q}^{\vec{n}} \in \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ with $\tilde{Q}^{\vec{n}} \prec Q^{\vec{n}}$. We show that $\tilde{Q}^{\vec{n}} \in \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ so that $\tilde{Q}^{\vec{n}} \in \mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$.

Note that $Q^{\vec{n}}$ is expressible as

$$\bigcurlyvee_{Q^{\vec{n}}(A_1,\ldots,A_p)=\mathrm{true}} \Bigg[\bigcurlywedge_{\langle\mathfrak{a}_1,\ldots,\mathfrak{a}_{n_1}\rangle\in A_1} H_1^{\mathfrak{a}_1,\ldots,\mathfrak{a}_{n_1}} \curlywedge \bigcurlywedge_{\langle\mathfrak{a}_1,\ldots,\mathfrak{a}_{n_1}\rangle\notin A_1} {\sim}H_1^{\mathfrak{a}_1,\ldots,\mathfrak{a}_{n_1}} \curlywedge \ldots \curlywedge \bigcurlywedge_{\langle\mathfrak{a}_1,\ldots,\mathfrak{a}_{n_p}\rangle\in A_p} H_p^{\mathfrak{a}_1,\ldots,\mathfrak{a}_{n_p}} \curlywedge \bigcurlywedge_{\langle\mathfrak{a}_1,\ldots,\mathfrak{a}_{n_p}\rangle\notin A_p} {\sim}H_p^{\mathfrak{a}_1,\ldots,\mathfrak{a}_{n_p}} \Bigg]$$

and is hence in $\mathcal{H}^{\theta}_{\mathfrak{D},\Sigma}$. (By (1) the disjuncts here number fewer than θ, and by (2) each disjunction comprises fewer than θ conjuncts.) ⊣

THEOREM 6.7. *Let* $|\mathfrak{D}| \geq \theta$. *Then* $\mathrm{CQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle = \emptyset$.

PROOF. Suppose for the sake of contradiction that $Q^{\vec{n}} \in \mathrm{CQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ with $|\mathfrak{D}| \geq \theta$. Then $Q^{\vec{n}} \in \mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle \cap \mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$. Since $\mathsf{false}^{\vec{n}}, \mathsf{true}^{\vec{n}} \notin \mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle$, it follows that $Q^{\vec{n}}$ is contingent. By Theorem 6.13 we have $|\mathfrak{D}| < \theta$, which is a contradiction. ⊣

THEOREM 6.8 (Theorem 6.7 of SPLQ). *Suppose that* $Q^{\vec{n}}$ *is a categorical quantifier and that* $\tilde{Q}^{\vec{n}}$ *is a logical quantifier. Then either* $Q^{\vec{n}}$ *and* $\tilde{Q}^{\vec{n}}$ *are inconsistent or* $Q^{\vec{n}} \preceq \tilde{Q}^{\vec{n}}$ (cf. Theorem 4.7 of SPLQ).

PROOF. Fixed-domain quantifier $Q^{\vec{n}} \curlywedge \tilde{Q}^{\vec{n}}$ is logical since both $Q^{\vec{n}}$ and $\tilde{Q}^{\vec{n}}$ are. So suppose that $Q^{\vec{n}} \curlywedge \tilde{Q}^{\vec{n}} \neq \mathsf{false}^{\vec{n}}$. Further, suppose, for the sake of proving a contradiction, that $Q^{\vec{n}} \not\preceq \tilde{Q}^{\vec{n}}$. Then satisfiable $Q^{\vec{n}} \curlywedge \tilde{Q}^{\vec{n}}$ is a logical quantifier satisfying $Q^{\vec{n}} \curlywedge \tilde{Q}^{\vec{n}} \prec Q^{\vec{n}}$, which contradicts $Q^{\vec{n}}$ categorical. We conclude that $Q^{\vec{n}} \curlywedge \tilde{Q}^{\vec{n}} \neq \mathsf{false}^{\vec{n}}$ implies $Q^{\vec{n}} \preceq \tilde{Q}^{\vec{n}}$. ⊣

THEOREM 6.9 (essentially Theorem 6.9 of SPLQ). *Let domain* $\mathfrak{D}$ *be given with* $|\mathfrak{D}| > 1$.

1. *Suppose that* $Q^{\vec{n}}$ *and* $\tilde{Q}^{\vec{n}}$ *are categorical quantifiers over* $\mathfrak{D}$ *and* Σ *with* $Q^{\vec{n}} \neq \tilde{Q}^{\vec{n}}$. *Then we have that* ${\sim}Q^{\vec{n}}$ *and* $Q^{\vec{n}} \curlyvee \tilde{Q}^{\vec{n}}$ *are logical but not categorical, and similarly for unsatisfiable* $Q^{\vec{n}} \curlywedge \tilde{Q}^{\vec{n}}$.
2. *Suppose that* $\{Q^{\vec{n}}_\alpha\}_{\alpha<\gamma}$ *with* $\gamma < \theta$ *is a family of categorical quantifiers over* $\mathfrak{D}$ *at least two of which are distinct. Then both* $\bigcurlyvee\{Q^{\vec{n}}_\alpha\}_{\alpha<\gamma}$ *and unsatisfiable* $\bigcurlywedge\{Q^{\vec{n}}_\alpha\}_{\alpha<\gamma}$ *are logical quantifiers over* $\mathfrak{D}$ *that are not categorical.*

Proof. For simplicity of exposition we assume $p = 2$. Suppose that $Q^{\vec{n}}$ and $\tilde{Q}^{\vec{n}}$ are distinct categorical quantifiers over $\mathfrak{D}$. Since both are logical, so is their join. Since $Q^{\vec{n}}, \tilde{Q}^{\vec{n}} \prec Q^{\vec{n}} \curlyvee \tilde{Q}^{\vec{n}}$, the latter is not categorical. (The proof that $\bigvee\{Q^{\vec{n}}_\alpha\}_{\alpha<\gamma}$ is logical but not categorical is essentially the same.)

Since $Q^{\vec{n}}$ and $\tilde{Q}^{\vec{n}}$ are both categorical and hence quasi-unitary, we can write $Q^{[A,B]}$ for the former and $Q^{[\tilde{A},\tilde{B}]}$ for the latter with $A, \tilde{A} \in \wp(\mathfrak{D}^{n_1})$ and $B, \tilde{B} \in \wp(\mathfrak{D}^{n_2})$. Further, that $Q^{\vec{n}} \neq \tilde{Q}^{\vec{n}}$ implies that for no $\pi \in S_{\mathfrak{D}}$ do we have both $\pi(A) = \tilde{A}$ and $\pi(B) = \tilde{B}$. By elementary reasoning regarding permutations, it follows that $Q^{\vec{n}} \curlywedge \tilde{Q}^{\vec{n}}$ is unsatisfiable and hence logical but not categorical. (Again, the demonstration that $\bigwedge\{Q^{\vec{n}}_\alpha\}_{\alpha<\gamma}$ is logical but not categorical is not different.)

In the minimal case such that $|\mathfrak{D}| = 2$ and $n_1 = n_2 = 1$ there are already ten satisfiable quasi-unitary quantifiers over $\mathfrak{D}$. If $Q^{\vec{n}}$ is one of them, then $\sim Q^{\vec{n}}$ is the join of the remaining nine and is hence logical but not categorical by the foregoing. ⊣

Lemma 6.10 (Lemma 6.10 of SPLQ). *Let domain $\mathfrak{D}$ be given. Then any permutation-invariant subset of $\wp(\mathfrak{D}^{n_1}) \times \ldots \times \wp(\mathfrak{D}^{n_p})$ is the (disjoint) union of minimal invariant subsets of $\wp(\mathfrak{D}^{n_1}) \times \ldots \times \wp(\mathfrak{D}^{n_p})$.*

Proof. If permutation-invariant $\mathfrak{C} \subseteq \wp(\mathfrak{D}^{n_1}) \times \ldots \times \wp(\mathfrak{D}^{n_p})$ is empty, then the result is obvious. So suppose that $\emptyset \neq \mathfrak{C} \subseteq \wp(\mathfrak{D}^{n_1}) \times \ldots \times \wp(\mathfrak{D}^{n_p})$ is permutation-invariant under the action of $S_{\mathfrak{D}}$. In that case, $S_{\mathfrak{D}}$ induces an equivalence relation on $\wp(\mathfrak{D}^{n_1}) \times \ldots \times \wp(\mathfrak{D}^{n_p})$ given by $\langle A_1, \ldots A_p\rangle \simeq \langle B_1, \ldots, B_p\rangle \Longleftrightarrow_{\text{def.}} \exists\pi \in S_{\mathfrak{D}}(\pi(A_1) = B_1 \wedge \ldots \wedge \pi(A_p) = B_p)$. Equivalence classes $[\langle A_1, \ldots, A_p\rangle]_{\simeq}$ are all minimal invariant, and $\mathfrak{C}$ is their union. ⊣

As an immediate consequence of Lemma 6.10 we have

Theorem 6.11 (essentially Theorem 6.11 of SPLQ). *Let domain $\mathfrak{D}$ be given. Then every logical quantifier $Q^{\vec{n}}$ is the join of a (possibly empty) set of categorical type-$\langle\vec{n}\rangle$ quantifiers.*

Proof. If $Q^{\vec{n}}$ is unsatisfiable, then it is $\bigvee\emptyset$. So assume that $Q^{\vec{n}}$ is satisfiable. So $\mathfrak{C} =: \{\langle A_1, \ldots, A_p\rangle \in \wp(\mathfrak{D}^{n_1}) \times \ldots \times \wp(\mathfrak{D}^{n_p}) \mid Q^{\vec{n}}(A_1, \ldots, A_p) = \text{true}\}$ is nonempty and permutation-invariant. We have by Lemma 6.10 that $\mathfrak{C}$ is the disjoint union of a family $\{\mathfrak{C}_\alpha\}_{\alpha<\gamma}$ of minimal invariant subsets of $\wp(\mathfrak{D}^{n_1}) \times \ldots \times \wp(\mathfrak{D}^{n_p})$. Moreover, by Theorem 6.3, each member $\mathfrak{C}_\alpha$ of this family is $\{\langle A_1, \ldots, A_p\rangle \in \wp(\mathfrak{D}^{n_1}) \times \ldots \times \wp(\mathfrak{D}^{n_p}) \mid Q^{\vec{n}}_\alpha(A_1, \ldots, A_p) = \text{true}\}$ for some quasi-unitary and hence categorical quantifier $Q^{\vec{n}}_\alpha$. So $Q^{\vec{n}}$ is $\bigvee_{\alpha<\gamma} Q^{\vec{n}}_\alpha$. ⊣

We justify the remark immediately preceding Representation Theorem 6.12 of SPLQ. Suppose that $\mathcal{K} \subseteq \mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ with $|\mathcal{K}| < \theta$. Since $\mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle \subseteq \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$, we have that $\bigvee\mathcal{K}, \bigwedge\mathcal{K} \in \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ by Theorem 6.3 of SPLQ. But $\mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle \subseteq \mathfrak{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ as well. Moreover, by the definition of $\mathfrak{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ we have that $\bigvee\mathcal{K}, \bigwedge\mathcal{K} \in \mathfrak{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$. It follows that $\bigvee\mathcal{K}, \bigwedge\mathcal{K} \in \mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle = \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle \cap \mathfrak{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$, which means that $\left\langle \mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle, \bigvee, \bigwedge, \sim, \mathsf{false}^{\vec{n}}, \mathsf{true}^{\vec{n}} \right\rangle$ is a θ-complete subalgebra of Boolean algebra $\left\langle \mathfrak{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle, \bigvee, \bigwedge, \sim, \mathsf{false}^{\vec{n}}, \mathsf{true}^{\vec{n}} \right\rangle$.

Assume that $|\mathfrak{D}| < \theta$. By definition any $Q^{\vec{n}} \in \mathrm{CQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ is a satisfiable member of $\mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ and such that any satisfiable quantifier $\tilde{Q}^{\vec{n}} \prec Q^{\vec{n}}$ is nonlogical and hence not in $\mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$. This is to say that $Q^{\vec{n}}$ is an atom of subalgebra $\big\langle \mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle, \curlyvee, \curlywedge, \sim, \mathsf{false}^{\vec{n}}, \mathsf{true}^{\vec{n}}\big\rangle$. That this subalgebra is atomic then follows from the proof of Theorem 6.11, wherein it is clear that if $Q^{\vec{n}}$ is satisfiable then $Q^{\vec{n}}$ is $\bigcurlyvee \mathcal{K}$ for some nonempty $\mathcal{K} \subseteq \mathrm{CQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$. (We have $\tilde{Q}^{\vec{n}} \preceq Q^{\vec{n}}$ for each $\tilde{Q}^{\vec{n}} \in \mathcal{K}$.)

On the other hand, if $|\mathfrak{D}| \geq \theta$, then, as is shown below, $\mathrm{LQ}^{\vec{n}}_{\mathfrak{D}}\langle\vec{n}\rangle$ contains no contingent quantifiers. Subalgebra $\big\langle \mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle, \curlyvee, \curlywedge, \sim, \mathsf{false}^{\vec{n}}, \mathsf{true}^{\vec{n}}\big\rangle$ is hence trivial.

The logic version of Representation Theorem 6.12 presented in SPLQ has a system version as consequence.

Theorem 6.12 (Second Representation Theorem (system version)). *Suppose given domain $\mathfrak{D}$ and signature $\Sigma = \langle R_1, \dots, R_p\rangle$. Then any $Q^{\vec{n}} \in \mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ is the quantifier image $Q^{\vec{n}}_{\mathcal{A}}$ of some $\mathcal{A} \in \mathrm{Cat}_{\mathfrak{D},\Sigma}$.*

Proof. Suppose $Q^{\vec{n}} \in \mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle$. By the remark following Definition 5.1 of SPLQ there exists strongly inaccessible θ such that $Q^{\vec{n}} \in \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$. It follows by Representation Theorem 6.12 of SPLQ that $Q^{\vec{n}} \in \mathrm{CQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle = \mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle \cap \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ is the quantifier image $Q^{\vec{n}}_{\mathcal{A}}$ of some $\mathcal{A} \in \mathrm{Cat}^{\theta}_{\mathfrak{D},\Sigma} \subseteq \mathrm{Cat}_{\mathfrak{D},\Sigma}$. ⊣

Representation Theorems 6.12 and 6.13 of SPLQ both make use of the following result.

Theorem 6.13. *If any member of $\mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ is contingent, then $|\mathfrak{D}| < \theta$.*

The reader should note the analogy with Theorem 4.18. Next, on analogy with Theorem 4.13 and recalling Definition 5.2, we have

Theorem 6.14. *Let $Q^{\vec{n}} \in \mathfrak{Q}_{\mathfrak{D}}\langle\vec{n}\rangle$. Suppose $Q^{\vec{n}}(A_1, \dots, A_p) = \mathrm{true}$, where $A_\ell \in \wp(\mathfrak{D}^{n_\ell})$ for all $1 \leq \ell \leq p$. Consider $B_1, \dots, B_p$ with $B_\ell \in \wp(\mathfrak{D}^{n_\ell})$ for all $1 \leq \ell \leq p$. Suppose that, for all $\mathfrak{a} \in \mathfrak{D}$ and all $1 \leq \ell \leq p$, we have that $\mathfrak{a} \in A_\ell \oplus B_\ell$ implies $\mathfrak{a} \notin \mathrm{Constit}(Q^{\vec{n}})$. Then $Q^{\vec{n}}(B_1, \dots, B_p) = \mathrm{true}$.*

Theorem 6.13 now follows and has as consequence that if $|\mathfrak{D}| \geq \theta$ then $\mathrm{CQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle = \emptyset$ and $\mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle = \{\mathsf{false}^{\vec{n}}, \mathsf{true}^{\vec{n}}\}$.

The logic version of Representation Theorem 6.13 presented in SPLQ has the following system version as consequence.

Theorem 6.15 (Third Representation Theorem (system version)). *Suppose given domain $\mathfrak{D}$ and signature Σ. Then any $Q^{\vec{n}} \in \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ is the quantifier image $Q^{\vec{n}}_{\mathcal{A}}$ of some $\mathcal{A} \in \mathrm{Sym}_{\mathfrak{D},\Sigma}$.*

Proof. Suppose $Q^{\vec{n}} \in \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$. By the remark following Definition 5.1 of SPLQ there exists strongly inaccessible θ such that $Q^{\vec{n}} \in \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$. It follows by Representation Theorem 6.13 of SPLQ that $Q^{\vec{n}} \in \mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle = \mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle \cap \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle\vec{n}\rangle$ is the quantifier image $Q^{\vec{n}}_{\mathcal{A}}$ of some $\mathcal{A} \in \mathrm{Sym}^{\theta}_{\mathfrak{D},\Sigma} \subseteq \mathrm{Sym}_{\mathfrak{D},\Sigma}$. ⊣

6.1. Concerning an earlier error now corrected. In an earlier version of SPLQ a certain footnote read as follows.

> By extension, Mostowski's (limited) generalized quantifiers are all of type $\langle n_1, \dots, n_p\rangle$ with $n_1 = \dots = n_p$ (see [10], Chapter 2). It is easy

> to extend this idea, however. For simplicity, we consider the case $p = 2$ with $n_1 = 1$ and $n_2 = 2$ only. Let $T : \{\langle \nu_1, \nu_2, \nu_3, \nu_4 \rangle \mid \nu_1 + \nu_2 + \nu_3 + \nu_4 = |\mathfrak{D}^2|\} \mapsto \{\text{false}, \text{true}\}$. Then we say that $Q_T^{1,2} : \wp(\mathfrak{D}^1) \times \wp(\mathfrak{D}^2) \mapsto \{\text{false}, \text{true}\}$ is a *generalized quantifier over* $\mathfrak{D}$ *in the extended sense*, where, for any $A \subseteq \mathfrak{D}^1$ and any $B \subseteq \mathfrak{D}^2$, we have that $Q_T^{1,2}(A, B) = T(|(A \times \mathfrak{D}) \cap B|, |(A \times \mathfrak{D}) \setminus B|, |B \setminus (A \times \mathfrak{D})|, |\mathfrak{D}^2 \setminus ((A \times \mathfrak{D}) \cup B)|)$.

In the body of the paper we stated that our notion of logical quantifier coincides with this notion of "generalized quantifier in the extended sense." This is wrong, as can be seen by reflection on categorical proposition $\mathcal{A}_3$ at the end of §1 of [11]. (That proposition expresses the fact that dyadic relation R linearly orders domain $\mathfrak{D}$.)

Justification for the erroneous equivalence took the form of Theorem 6.16, which in turn relied upon likewise erroneous Lemma 6.17. For simplicity we considered type-$\langle 1, 2\rangle$ quantifiers only.

Theorem 6.16 (erroneous). *Quantifier* $Q^{1,2} \in \mathrm{LQ}_{\mathfrak{D}}\langle 1, 2\rangle$ *if and only if* $Q^{1,2} = Q_T^{1,2}$ *for some* $T : \{\langle \nu_1, \nu_2, \nu_3, \nu_4 \rangle \mid \nu_1 + \nu_2 + \nu_3 + \nu_4 = |\mathfrak{D}^2|\} \mapsto \{\text{false}, \text{true}\}$.

Proof. For the one direction, suppose that $Q^{1,2} = Q_T^{1,2}$. Then, for any $A \subseteq \mathfrak{D}$, any $B \subseteq \mathfrak{D}^2$, and any $\pi \in S_{\mathfrak{D}}$, we can write

$$Q^{1,2}(A, B) = \text{true}$$

$$\Longleftrightarrow T(|(A \times \mathfrak{D}) \cap B|, |(A \times \mathfrak{D}) \setminus B|, |B \setminus (A \times \mathfrak{D})|, |\mathfrak{D}^2 \setminus ((A \times \mathfrak{D}) \cup B)|) = \text{true}$$

since $Q^{1,2} = Q_T^{1,2}$

$$\Longleftrightarrow T(|(\pi(A) \times \mathfrak{D}) \cap \pi(B)|, |(\pi(A) \times \mathfrak{D}) \setminus \pi(B)|, \\ |\pi(B) \setminus (\pi(A) \times \mathfrak{D})|, |\mathfrak{D}^2 \setminus ((\pi(A) \times \mathfrak{D}) \cup \pi(B))|) = \text{true}$$

since any domain permutation preserves cardinalities

$$\Longleftrightarrow Q_T^{1,2}(\pi(A), \pi(B)) = \text{true}$$
$$\Longleftrightarrow Q^{1,2}(\pi(A), \pi(B)) = \text{true}$$

since $Q_T^{1,2} = Q^{1,2}$.

For the other direction, suppose that $Q^{1,2} \in \mathrm{LQ}_{\mathfrak{D}}\langle 1, 2\rangle$. By Theorem 6.11 we have that $Q^{1,2}$ is $\curlyvee \mathcal{J}$ for some $\mathcal{J} \subseteq \mathrm{CQ}_{\mathfrak{D}}\langle 1, 2\rangle$. But each member of $\mathcal{J}$ is a quasi-unitary quantifier and hence a generalized quantifier in the extended sense by Lemma 6.17. So $Q^{1,2}$ is a join of generalized quantifiers in the extended sense. Finally, any join of a set of extended quantifiers in the extended sense is a generalized quantifier in the extended sense, and we are done. ⊣

The problem rests with the proof of

Lemma 6.17 (erroneous). *Any quasi-unitary type-*$\langle 1, 2\rangle$ *quantifier is a type-*$\langle 1, 2\rangle$ *generalized quantifier in the extended sense.*

Proof. Suppose that $Q^{1,2}$ is quasi-unitary. Then there exist $A_1 \subseteq \mathfrak{D}$ and $A_2 \subseteq \mathfrak{D}^2$ with $Q^{1,2} = Q^{[A_1,A_2]}$. We define

$$\begin{aligned}
\kappa_1 &=_{\text{def.}} |(A_1 \times \mathfrak{D}) \cap A_2| \\
\kappa_2 &=_{\text{def.}} |(A_1 \times \mathfrak{D}) \setminus A_2| \\
\kappa_3 &=_{\text{def.}} |A_2 \setminus (A_1 \times \mathfrak{D})| \\
\kappa_4 &=_{\text{def.}} |\mathfrak{D}^2 \setminus ((A_1 \times \mathfrak{D}) \cup A_2)|.
\end{aligned}$$

Further, let $T : \{\langle \nu_1, \nu_2, \nu_3, \nu_4 \rangle \mid \nu_1 + \nu_2 + \nu_3 + \nu_4 = |\mathfrak{D}^2|\} \mapsto \{\text{false}, \text{true}\}$ be defined by $T(\nu_1, \nu_2, \nu_3, \nu_4) = \text{true} \iff_{\text{def.}} \nu_1 = \kappa_1$ and $\nu_2 = \kappa_2$ and $\nu_3 = \kappa_3$ and $\nu_4 = \kappa_4$. For arbitrary $B_1 \subseteq \mathfrak{D}$ and arbitrary $B_2 \subseteq \mathfrak{D}^2$ we have

$$\begin{aligned}
&Q^{1,2}(B_1, B_2) = \text{true} \\
\iff &B_1 = \pi(A_1) \text{ and } B_2 = \pi(A_2) \text{ for some } \pi \in S_{\mathfrak{D}}
\end{aligned}$$

since $Q^{1,2}$ is $Q^{[A_1,A_2]}$ by assumption

$$\begin{aligned}
\iff &B_1 \times \mathfrak{D} = \pi(A_1) \times \mathfrak{D} \text{ and } B_2 = \pi(A_2) \text{ for some } \pi \in S_{\mathfrak{D}} \\
\iff &B_1 \times \mathfrak{D} = \pi(A_1) \times \pi(\mathfrak{D}) \text{ and } B_2 = \pi(A_2) \text{ for some } \pi \in S_{\mathfrak{D}}
\end{aligned}$$

since any permutation fixes $\mathfrak{D}$ setwise

$$\iff B_1 \times \mathfrak{D} = \pi(A_1 \times \mathfrak{D}) \text{ and } B_2 = \pi(A_2) \text{ for some } \pi \in S_{\mathfrak{D}}$$

since $\pi(\langle \mathfrak{a}, \mathfrak{b} \rangle) = \langle \pi(\mathfrak{a}), \pi(\mathfrak{b}) \rangle$ for any $\mathfrak{a} \in A_1$ and any $\mathfrak{b} \in \mathfrak{D}$

$$\begin{aligned}
\iff &(B_1 \times \mathfrak{D}) \cap B_2 = \pi(A_1 \times \mathfrak{D}) \cap \pi(A_2) \text{ and} \\
&\qquad (B_1 \times \mathfrak{D}) \setminus B_2 = \pi(A_1 \times \mathfrak{D}) \setminus \pi(A_2) \text{ and} \\
&B_2 \setminus (B_1 \times \mathfrak{D}) = \pi(A_2) \setminus \pi(A_1 \times \mathfrak{D}) \text{ and} \\
&\qquad \mathfrak{D}^2 \setminus ((B_1 \times \mathfrak{D}) \cup B_2) = \mathfrak{D}^2 \setminus (\pi(A_1 \times \mathfrak{D}) \cup \pi(A_2)) \\
\iff &|(B_1 \times \mathfrak{D}) \cap B_2| = |\pi(A_1 \times \mathfrak{D}) \cap \pi(A_2)| \text{ and} \\
&\qquad |(B_1 \times \mathfrak{D}) \setminus B_2| = |\pi(A_1 \times \mathfrak{D}) \setminus \pi(A_2)| \text{ and} \\
&|B_2 \setminus (B_1 \times \mathfrak{D})| = |\pi(A_2) \setminus \pi(A_1 \times \mathfrak{D})| \text{ and} \\
&\qquad |\mathfrak{D}^2 \setminus ((B_1 \times \mathfrak{D}) \cup B_2)| = |\mathfrak{D}^2 \setminus (\pi(A_1 \times \mathfrak{D}) \cup \pi(A_2))|
\end{aligned}$$

Error: only the forward direction holds here since from the fact that two sets have the same cardinality one cannot infer that they are identical.

$$\begin{aligned}
\iff &|(B_1 \times \mathfrak{D}) \cap B_2| = |(A_1 \times \mathfrak{D}) \cap A_2| \text{ and} \\
&\qquad |(B_1 \times \mathfrak{D}) \setminus B_2| = |(A_1 \times \mathfrak{D}) \setminus A_2| \text{ and} \\
&|B_2 \setminus (B_1 \times \mathfrak{D})| = |A_2 \setminus (A_1 \times \mathfrak{D})| \text{ and} \\
&\qquad |\mathfrak{D}^2 \setminus ((B_1 \times \mathfrak{D}) \cup B_2)| = |\mathfrak{D}^2 \setminus ((A_1 \times \mathfrak{D}) \cup A_2)|
\end{aligned}$$

since any permutation preserves cardinalities

$$\begin{aligned}
\Longleftrightarrow & |(B_1 \times \mathfrak{D}) \cap B_2| = \kappa_1 \text{ and } |(B_1 \times \mathfrak{D}) \setminus B_2| = \kappa_2 \text{ and} \\
& \qquad |B_2 \setminus (B_1 \times \mathfrak{D})| = \kappa_3 \text{ and } |\mathfrak{D}^2 \setminus ((B_1 \times \mathfrak{D}) \cup B_2)| = \kappa_4 \\
\Longleftrightarrow & T(|(B_1 \times \mathfrak{D}) \cap B_2|, |(B_1 \times \mathfrak{D}) \setminus B_2|, \\
& \qquad |B_2 \setminus (B_1 \times \mathfrak{D})|, |\mathfrak{D}^2 \setminus ((B_1 \times \mathfrak{D}) \cup B_2)|) = \text{true} \\
\Longleftrightarrow & Q_T^{1,2}(B_1, B_2) = \text{true}
\end{aligned}$$

by the definition of $Q_T^{1,2}$.

⊣

Our confusion was the result of linking Mostowski's ideas too closely with the notion of cardinality. Generally, logical quantifiers in our sense (and that of Lindström) concern not cardinality but, rather, structure. The value of this discussion seems to be the following. Mostowski considered only (unary) predicative quantifiers, by which we mean those of type $\langle 1 \rangle$. In that restricted setting, structure coincides with cardinality. One can extend Mostowski's discussion to relational quantifiers in either of two ways. One way to do this, emphasizing cardinality, was described in the footnote quoted above. That approach yields but a proper subset of logical quantifiers as defined in Definition 6.1 of SPLQ. (Our linear-ordering quantifier is not in that subset.) Usually Mostowski's notion of limited generalized quantifier is extended in the manner of Lindström's [6], in which case the extended notion of generalized quantifier coincides with our notion of logical quantifier over fixed domain.

6.2. Classes versus sets. It is true that, whatever the cardinality of domain $\mathfrak{D}$, the cardinality of $\mathfrak{Q}_{\mathfrak{D}}\langle \vec{n} \rangle$ is strictly less than the class of all ordinals. But the ordinals do not play the role in the definition of quantifier algebras that they play in the definition of Zermelo systems and Zermelo logics (cf. §4.2). Rather, $\mathfrak{Q}_{\mathfrak{D}}\langle \vec{n} \rangle$ itself serves as the universe in the former context. In light of this we say that $\mathfrak{Q}_{\mathfrak{D}}\langle \vec{n} \rangle$ is a *class*. The atoms of algebra $\langle \mathrm{LQ}_{\mathfrak{D}}\langle \vec{n} \rangle, \curlyvee, \curlywedge, \sim, \mathsf{false}^{\vec{n}}, \mathsf{true}^{\vec{n}} \rangle$ are quasi-unitary (or categorical) quantifiers as described following Definition 6.6 of SPLQ. Each member of $\mathrm{LQ}_{\mathfrak{D}}\langle \vec{n} \rangle \subseteq \mathfrak{Q}_{\mathfrak{D}}\langle \vec{n} \rangle$ is the join of a (possibly empty) subset of the collection of atoms. It follows by Cantor's Theorem that the atoms constitute a set. That is, $\mathrm{CQ}_{\mathfrak{D}}\langle \vec{n} \rangle$ is a set.

As for $\mathrm{LQ}_{\mathfrak{D}}\langle \vec{n} \rangle$, we reason as follows. First, the cardinality of $\mathfrak{Q}_{\mathfrak{D}}\langle \vec{n} \rangle$ is

$$\nu^* =: 2^{2^{|\mathfrak{D}|^{n_1}} \cdot \ldots \cdot 2^{|\mathfrak{D}|^{n_p}}}.$$

The question is now, How many of the ν^* quantifiers in $\mathfrak{Q}_{\mathfrak{D}}\langle \vec{n} \rangle$ are permutation-invariant? Writing κ^* for that number, we then ask, Is $\kappa^* < \nu^*$? If so, then $\mathrm{LQ}_{\mathfrak{D}}\langle \vec{n} \rangle$ is a set. On the other hand, if $\kappa^* = \nu^*$, then $\mathrm{LQ}_{\mathfrak{D}}\langle \vec{n} \rangle$ is a class. We have three cases to consider:

1. If $|\mathfrak{D}| = 1$ then $\mathrm{LQ}_{\mathfrak{D}}\langle \vec{n} \rangle = \mathfrak{Q}_{\mathfrak{D}}\langle \vec{n} \rangle$ is a class.

2. If $\mathfrak{D}$ is finite and contains at least two elements, then κ^* finite is strictly less than ν^* finite. (At least one member of $\mathcal{Q}_{\mathfrak{D}}\langle\vec{n}\rangle$ is not permutation-invariant.) So $\mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ is a set.
3. If $|\mathfrak{D}| = \aleph_\alpha$, then $\nu^* = \aleph_{\alpha+2}$. We exploit the relation with Mostowski's cardinality quantifiers. It is sufficient to consider type-$\langle 1,1\rangle$ cardinality quantifiers. By footnote 1 of SPLQ we need only count cardinal quadruples $\langle\nu_1,\nu_2,\nu_3,\nu_4\rangle$ satisfying $\nu_1+\nu_2+\nu_3+\nu_4 = |\mathfrak{D}|$. There are three cases to consider:
   a. If $\alpha = 0$, then $|\mathrm{CQ}_{\mathfrak{D}}\langle 1,1\rangle| = 4\cdot(\aleph_0+1)\cdot(\aleph_0+1)\cdot(\aleph_0+1) = \aleph_0$. So $|\mathrm{LQ}_{\mathfrak{D}}\langle 1,1\rangle| = \aleph_1$. Since $\nu^* = \aleph_2$, we see that $\mathrm{LQ}_{\mathfrak{D}}\langle 1,1\rangle$ is a set.
   b. If $\alpha = \beta+1$, then $|\mathrm{CQ}_{\mathfrak{D}}\langle 1,1\rangle| = 4\cdot((\overline{\overline{\alpha}}+1)+\aleph_0)\cdot((\overline{\overline{\alpha}}+1)+\aleph_0)\cdot((\overline{\overline{\alpha}}+1)+\aleph_0) < \aleph_\alpha$. So $|\mathrm{LQ}_{\mathfrak{D}}\langle 1,1\rangle| < \aleph_{\alpha+1}$. Since $\nu^* = \aleph_{\alpha+2}$, we see that $\mathrm{LQ}_{\mathfrak{D}}\langle 1,1\rangle$ is a set.
   c. If α is a limit, then $|\mathrm{CQ}_{\mathfrak{D}}\langle 1,1\rangle| = 4\cdot((\overline{\overline{\alpha}}+1)+\aleph_0)\cdot((\overline{\overline{\alpha}}+1)+\aleph_0)\cdot((\overline{\overline{\alpha}}+1)+\aleph_0) \le \aleph_\alpha$. So $|\mathrm{LQ}_{\mathfrak{D}}\langle 1,1\rangle| \le \aleph_{\alpha+1}$. Since $\nu^* = \aleph_{\alpha+2}$, we see that $\mathrm{LQ}_{\mathfrak{D}}\langle 1,1\rangle$ is a set.

Turning to $\mathcal{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$, it seems reasonable to let that collection now play the role of universe. If $|\mathfrak{D}| < \theta$, the atoms of algebra $\langle \mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle, \curlyvee, \curlywedge, \sim, \mathsf{false}^{\vec{n}}, \mathsf{true}^{\vec{n}}\rangle$ are quasi-unitary (or categorical) quantifiers as described following Definition 6.6 of SPLQ. Each member of $\mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle \subseteq \mathcal{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ is the join of a (possibly empty) subset of the collection of atoms. It follows by Cantor's Theorem that the atoms constitute a set. That is, $\mathrm{CQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ is a set. If $|\mathfrak{D}| \ge \theta$, then $\mathrm{CQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle = \emptyset$ is a set.

As for $\mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$, we have three cases to consider:

1. If $|\mathfrak{D}| = 1$, then $\mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle = \mathcal{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ is a class.
2. If $\mathfrak{D}$ is finite and contains at least two elements, then both $\mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ and $\mathcal{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ are finite, and at least one member of the latter is not permutation-invariant. So $\mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ is a set.
3. If $\mathfrak{D}$ is infinite with $|\mathfrak{D}| < \theta$, then there are three cases to consider. We can reason as before, and $\mathrm{LQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ is a set. If $|\mathfrak{D}| \ge \theta$, then $\mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle = \mathcal{Q}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle = \{\mathrm{false}^{\vec{n}}, \mathrm{true}^{\vec{n}}\}$. Hence $\mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ is a class.

In summary, since in the cases to which we give precedence both $\mathrm{CQ}_{\mathfrak{D}}\langle\vec{n}\rangle$ ($\mathrm{CQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$) and $\mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$ ($\mathrm{LQ}^\theta_{\mathfrak{D}}\langle\vec{n}\rangle$) are sets, we shall say generally that they are sets.

§7. Concluding philosophical remarks.

We give two more examples along the lines of footnote 3 of SPLQ. Negation is the quantifier image $\sim H_1^{\langle\,\rangle} \in \mathcal{Q}^\theta_{\mathfrak{D}}\langle 0\rangle$ of symmetric $\neg R\ldots \in \mathcal{H}^\theta_{\mathfrak{D},\Sigma}$ having model set $\{\langle\mathbf{0}\rangle\} \subseteq \mathfrak{T}^0_{\mathfrak{D}}$. (Our writing $R\ldots$ is intended to suggest R zero-adic.) Frege's assertion, on the other hand, is $H_1^{\langle\,\rangle}$, which is the quantifier image of symmetric $R\ldots \in \mathcal{H}^\theta_{\mathfrak{D},\Sigma}$ having model set $\{\langle\mathbf{1}\rangle\}$.

Regarding the parenthetical remark at the end of the penultimate paragraph, it might be countered that Zermelo's two examples reveal nothing regarding his intentions since they are merely the simplest symmetric propositions available. However, that is not really true: tautology $R\mathfrak{a}\vee\neg R\mathfrak{a}$ and contradiction $R\mathfrak{a}\wedge\neg R\mathfrak{a}$ are arguably simpler but are not mentioned by Zermelo. What he has provided are the simplest examples that he regards as *characteristic* of the notion of symmetric proposition.

We justify a claim in footnote 4 of SPLQ.

THEOREM 7.1. *Let domain $\mathfrak{D}$ be given and let θ be any strongly inaccessible cardinal with $\theta > |\mathfrak{D}|$. Then $\mathfrak{C}^{((j_1),\cdots,(j_k))} \subseteq \wp(\mathfrak{D}^{j_1}) \times \ldots \times \wp(\mathfrak{D}^{j_k})$ is logical in Tarski's sense just in case $\mathfrak{C}^{((j_1),\cdots,(j_k))}$ is $[Q^{j_1,\cdots,j_k}]^{-1}(\text{true})$ for some $Q^{j_1,\cdots,j_k} \in \mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle j_1,\ldots,j_k\rangle$.*

PROOF. Suppose that $\mathfrak{C}^{((j_1),\cdots,(j_k))} \subseteq \wp(\mathfrak{D}^{j_1})\times\ldots\times\wp(\mathfrak{D}^{j_k})$. We define $Q^{j_1\cdots,j_k}$ by writing

$$Q^{j_1,\cdots,j_k} : \wp(\mathfrak{D}^{j_1}) \times \ldots \times \wp(\mathfrak{D}^{j_k}) \mapsto \{\text{true}, \text{false}\}$$

$$Q^{j_1,\cdots,j_k}(A_1,\ldots,A_k) = \begin{cases} \text{true} & \text{if } \langle A_1,\ldots,A_k\rangle \in \mathfrak{C}^{((j_1),\cdots,(j_k))} \\ \text{false} & \text{otherwise.} \end{cases}$$

Since $|\mathfrak{D}| < \theta$ by hypothesis, it follows that $|\mathfrak{C}^{((j_1),\cdots,(j_k))}| \leq |\wp(\mathfrak{D}^{j_1}) \times \ldots \times \wp(\mathfrak{D}^{j_k})| < \theta$ also. So $Q^{j_1,\cdots,j_k}$ is the join of fewer than θ unitary type-$\langle j_1,\ldots,j_k\rangle$ quantifiers, one for each member of $\mathfrak{C}^{((j_1),\cdots,(j_k))}$. Since all such quantifiers are in $\mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle j_1,\ldots,j_k\rangle$ by the remark following Theorem 5.1, we have that $Q^{j_1,\cdots,j_k}$ is itself in $\mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle j_1,\ldots,j_k\rangle$ by θ-closure. Now we can write

$$\begin{aligned}
&\mathfrak{C}^{((j_1,\cdots,j_k))} \text{ is logical in Tarski's sense} \\
\Longleftrightarrow &\text{for any } \pi \in S_{\mathfrak{D}} \text{ and any } A_1 \subseteq \mathfrak{D}^{j_1},\ldots,A_k \subseteq \mathfrak{D}^{j_k}, \\
&\langle A_1,\ldots,A_k\rangle \in \mathfrak{C}^{((j_1),\cdots,(j_k))} \text{ implies } \langle \pi(A_1),\ldots,\pi(A_k)\rangle \in \mathfrak{C}^{((j_1),\cdots,(j_k))} \\
\Longleftrightarrow &\text{for any } \pi \in S_{\mathfrak{D}} \text{ and any } A_1 \subseteq \mathfrak{D}^{j_1},\ldots,A_k \subseteq \mathfrak{D}^{j_k}, \\
&Q^{j_1,\cdots,j_k}(A_1,\ldots,A_k) = \text{true implies } Q^{j_1,\cdots,j_k}(\pi(A_1),\ldots,\pi(A_k)) = \text{true} \\
\Longleftrightarrow &Q^{j_1,\cdots,j_k} \in \mathrm{LQ}^{\theta}_{\mathfrak{D}}\langle j_1,\ldots,j_k\rangle
\end{aligned}$$

since $Q^{j_1,\cdots,j_k} \in \mathfrak{Q}^{\theta}_{\mathfrak{D}}\langle j_1,\ldots,j_k\rangle$ is permutation-invariant and hence logical.

Moreover, for all $A_1 \in \wp(\mathfrak{D}^{j_1}),\ldots,A_k \in \wp(\mathfrak{D}^{j_k})$, we have

$$\begin{aligned}
&\langle A_1,\ldots,A_k\rangle \in \mathfrak{C}^{((j_1),\cdots,(j_k))} \\
\Longleftrightarrow &Q^{j_1,\cdots,j_k}(A_1,\ldots,A_k) = \text{true} \\
\Longleftrightarrow &\langle A_1,\ldots,A_k\rangle \in [Q^{j_1,\cdots,j_k}]^{-1}(\text{true})
\end{aligned}$$

⊣

Terms of type 0^j in Tarski's sense are mentioned in footnote 4 of SPLQ, but no attempt is made to spell out their relation to anything in Zermelo. Now it can be seen that each logical term of type 0^j over fixed domain $\mathfrak{D}$ corresponds to a symmetric proposition over arbitrary nonempty $\mathfrak{D}$ with $|\mathfrak{D}| \geq j$ and arbitrary signature Σ having at least one j-adic member. This is indicated in Tables 1–3 for cases $j = 0, 1, 2$, where, for simplicity only, we assume that $\mathfrak{D} = \{a, b\}$ and that Σ is a unit signature. Note that, if t is a logical term of type 0^j in the left column of Table j and $\mathcal{A}_t$ is the corresponding symmetric proposition in the

TABLE 1. Logical terms of type 0^0

Logical term	Corresponding symmetric proposition where $\mathfrak{D} = \{a, b\}$ and $\Sigma = \langle R \rangle$ with R zero-adic
$\{\langle\,\rangle\}$	$R \ldots$
$\emptyset$	$\neg R \ldots$

TABLE 2. Logical terms of type 0^1

Logical term	Corresponding symmetric proposition where $\mathfrak{D} = \{a, b\}$ and $\Sigma = \langle R \rangle$ with R monadic
$\{\langle a \rangle, \langle b \rangle\}$	$Ra \wedge Rb$
$\emptyset$	$\neg Ra \wedge \neg Rb$

TABLE 3. Logical terms of type 0^2

Logical term	Corresponding symmetric proposition where $\mathfrak{D} = \{a, b\}$ and $\Sigma = \langle R \rangle$ with R dyadic
$\{\langle a, a \rangle, \langle a, b \rangle, \langle b, a \rangle, \langle b, b \rangle\}$	$aRa \wedge aRb \wedge bRa \wedge bRb$
$\emptyset$	$\neg aRa \wedge \neg aRb \wedge \neg bRa \wedge \neg bRb$
$\{\langle a, a \rangle, \langle b, b \rangle\}$	$aRa \wedge \neg aRb \wedge \neg bRa \wedge bRb$
$\{\langle a, b \rangle, \langle b, a \rangle\}$	$\neg aRa \wedge aRb \wedge bRa \wedge \neg bRb$

right column, then $t = B \cap \mathfrak{D}^j$, where B is the unique member of $\wp(\mathfrak{D}^j)$ such that $Q^j_{\mathcal{A}_t}(\langle B \rangle) = \text{true}$ with unary $Q^j_{\mathcal{A}_t} \in \mathrm{LQ}^\theta_{\mathfrak{D}}\langle j \rangle$ by Representation Theorem 6.5 of SPLQ.

What is now called the Tarski–Sher thesis is the claim that logical terms *tout court* are those invariant under isomorphisms. Tarski's proposal is then the weaker claim that logical terms over a fixed domain are those invariant under domain permutations—weaker because Tarski–Sher implies Tarski but not vice versa. Despite this, or just because of this, arguments *contra* Tarski will surely be of interest even to those who believe Tarski–Sher: any evidence against the former also counts against the latter. The last paragraph of SPLQ presents an argument *pro* Tarski. To the extent that this argument increases our confidence that Tarski is not false, it would seem to also increase, albeit to a lesser degree, our confidence that the stronger Tarski–Sher is not false. The general point is that arguments for and against the weaker claim should yet be of interest to those whose focus is the stronger claim.

Regarding our Confluence of Ideas Argument for Tarski's proposal to the effect that logical terms over a fixed domain be identified with those invariant under domain permutations, we mention the possibility of Tarski's having directly influenced Zermelo. The latter's [14], containing the first description of his theory of symmetric propositions, was published in 1931. Just two years earlier Zermelo spent a few weeks at the University of Warsaw, where he gave a series of nine lectures. We do not know how much contact he had with the young Tarski—probably little. We know that the two did meet, however (see [3], p. 354). (We seem to recall a favorable remark regarding a meeting with Tarski in Zermelo's correspondence but cannot locate the reference.) Even if there was no direct influence, it is likely that domain permutations were in the air in Warsaw during 1929 and that Zermelo's thinking was affected. What evidence we have indicates that Zermelo's own Warsaw lectures did not mention domain permutations (see [13]). So this is probably not an idea he brought to Warsaw. But even if Zermelo was influenced directly or indirectly by Tarski or even if they were both influenced by someone else, the cited confluence should yet be of some interest. After all, Tarski's focus is the finitary formal languages derivable from *Principia* whereas Zermelo's hierarchies are nonlinguistic systems of propositions of infinite length wherein strongly inaccessible cardinals play a special role. Nonetheless, our Isomorphism Theorem shows that their ideas regarding the logical turn out to coincide.

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