SYSTEMS OF EQUATIONS AND MATRICES WITH THE TI-89

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INTRODUCTION TO SYSTEMS OF LINEAR EQUATIONS

When two linear equations with two unknowns are solved, there are three possible outcomes. These outcomes are illustrated by the following examples.

Example 1. (One unique solution exists) Solve

$$\begin{aligned} x + 2y &= 5 \\ 3x + 11y &= 5 \end{aligned}$$

Solution: Multiplying the first equation by -3 and adding it to the second equation we obtain

$$\begin{aligned} -3x - 6y &= -15 \\ \underline{3x + 11y} &= \underline{\;\;5} \\ 5y &= -10. \end{aligned}$$

Dividing by 5 we find $y = -2.$

Substituting $y = -2$ into $x + 2y = 5$ we get $x + 2(-2) = 5$

So that $x = 9.$

The answer is $\begin{aligned} x &= 9 \\ y &= -2 \end{aligned}$ which is usually written (9, -2)

Check: $x + 2y = 5$: $9 + 2(-2) = 9 - 4 = 5$ checks.

$3x + 11y = 5$: $3(9) + 11(-2) = 27 - 22 = 5$ checks.

Example 2. (Infinitely many solutions exist) Solve

$$\begin{aligned} x + 2y &= 5 \\ 3x + 6y &= 15 \end{aligned}$$

Solution: Multiplying the first equation by -3 and adding it to the second equation we obtain

$$\begin{aligned} -3x - 6y &= -15 \\ \underline{3x + 6y} &= \underline{\;15} \\ 0 &= 0 \end{aligned}$$

This signifies the fact that the second equation did not contain any information that was not already present in the first equation. The second equation is just 3 times the first equation. That is, the original system of equations is the same as the result

$$\begin{aligned} x + 2y &= 5 \\ 0 &= 0. \end{aligned}$$

So y can be assigned any value whatsoever and then $x = 5 - 2y$ provides the rest of the solution. There are infinitely many solutions given by $(5 - 2y, y)$ where y can be any real number.

Check: A partial check can be performed by assigning any value to y and then checking that $(5 - 2y, y)$ provides a solution. For example, if $y = 4$ then $(5 - 2y, y) = (5 - 2(4), 4) = (-3, 4)$. Checking this in the original equations we see:

$x + 2y = 5$: $-3 + 2(4) = -3 + 8 = 5$ checks.

$3x + 6y = 15$: $3(-3) + 6(4) = -9 + 24 = 15$ checks.

You should repeat this partial check using some other value of y.
The complete general check, however, is the best one. It consists of substituting $x = 5 - 2y$ into the original equations as follows:

$x + 2y = 5$: $(5 - 2y) + 2y = 5$ checks.
$3x + 6y = 15$: $3(5 - 2y) + 6y = 15 - 6y + 6y = 15$ checks.

Example 3. (No solution exists.) Solve

$$\begin{aligned} x + 2y &= 5 \\ 3x + 6y &= 16 \end{aligned}$$

Solution: Multiplying the first equation by -3 and adding it to the second equation we obtain

$$\begin{aligned} -3x - 6y &= -15 \\ \underline{3x + 6y} &\underline{= 16} \\ 0 &= 1 \end{aligned}$$

This signifies the fact that the second equation contradicts the information contained in the first equation. It is impossible for $3x + 6y$ to equal 15 (first equation) at the same time that it equals 16 (second equation).

Now, based on these examples, you might think that it is easy to see when an equation provides no additional information (as in example 2) or contradictory information (as in example 3). But for more than two variables this is often not evident at all. Consider the two equations

$$\begin{aligned} 3x - 5y + 2z &= 11 \\ 2x + 3y - 4z &= 1 \end{aligned}$$

If the first equation is multiplied by 4 and the second equation is multiplied by -3, we get as the sum:

$$\begin{aligned} 12x - 20y + 8z &= 44 \\ \underline{-6x - 9y + 12z} &\underline{= -3} \\ 6x - 29y + 20z &= 41. \end{aligned}$$

If you were given the system of equations consisting of the original two equations and the result just obtained,

$$\begin{aligned} 3x - 5y + 2z &= 11 \\ 2x + 3y - 4z &= 1 \\ 6x - 29y + 20z &= 41 \end{aligned}$$

would you be able to spot the fact that the last equation contained no information that was not already contained in the prior two equations so that the solution would lead to $0 = 0$ as in example 2? And if the 41 were changed to 42, would you realize that the last equation contradicts the prior two equations so that the solution would lead to $0 = 1$ as in example 3? In the following sections we will see how to solve systems of linear equations with more than two variables. The basic ideas are illustrated by the previous three examples.

Note that the three systems of equations in the examples produced the following equivalent systems that provided the desired answers:

Example 1:
$$\begin{aligned} x \quad &= 9 \\ y &= -2 \end{aligned}$$

Example 2:
$$\begin{aligned} x + 2y &= 5 \\ 0 &= 0 \end{aligned}$$

Example 3:
$$\begin{aligned} x + 2y &= 5 \\ 0 &= 1 \end{aligned}$$

There is a pattern here that every system of linear equations can be reduced to. Namely, either

each equation starts with a variable not in the other equations
or
the equation is 0 = 0
or
the equation is 0 = 1 (or some other nonzero number).

If 0 = 1 (or some other nonzero number appears), then there is no solution.
If every variable has a numerical value (and 0 = 1 does not appear), then there is one unique solution.
Otherwise, there are infinitely many solutions.

Note that 0 = 0 is simply ignored entirely.

The system of equations in example 1
$$\begin{aligned} x + 2y &= 5 \\ 3x + 11y &= 5 \end{aligned}$$

can be written in shorthand as the matrix
$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 11 & 5 \end{bmatrix}.$$

Similarly, the resulting equations in example 1
$$\begin{aligned} x \quad &= 9 \\ y &= -2 \end{aligned}$$

can be written as the matrix
$$\begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \end{bmatrix}$$

since $1x + 0y = 9$ is the same as $x = 9$ and $0x + 1y = -2$ is the same as $y = -2$.

Likewise, the result of example 2 can be written as $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

and the result of example 3 can be written as $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 1 \end{bmatrix}$.

When solving equations nothing further is usually done with this last result since it already indicates the fact that no solution exists (0 = 1). But it could be taken one step further by adding -5 times the second equation (0 = -5) to the first equation ($x + 2y = 5$) to obtain $x + 2y = 0$, or $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The resulting matrices share the following common features.

Reduced Row Echelon Form (rref) of a matrix:

Rows filled with 0 (if any) appear at the bottom of the matrix.
The first entry in each of the other rows is 1. Zeros appear above and below this leading 1.
The leading 1 s descend ladder like from left to right.

Fact: The matrix for a system of linear equations is equivalent to exactly one reduced row echelon form matrix.

Example 4: The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 15 \end{bmatrix} \quad \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 & 0 & 8 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & -2 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice the third matrix has a 1 in the second row and fourth column and has an 8 (not a 0) above it. This is permissible because it is not the leading 1 in the second row.

Example 5: The following matrices are not in reduced row echelon form for the reasons given.

$\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 4 \end{bmatrix}$ The leading 1s do not descend ladder like from left to right.

$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ The row filled with 0s is not at the bottom.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$ The 2 appearing above the leading 1 of the second row should be 0.

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & -1 & 7 \end{bmatrix}$$ The first entry in the second row is -1 instead of 1.

A system of equations is said to be **consistent** if it has at least one solution. Thus, a system that has either one solution or infinitely many solutions is said to be consistent. If a system of equations does not have a solution, then it is said to be **inconsistent**. Recalling the interpretation of the solutions of the first three examples (see page 3), the following holds.

INTERPRETATION OF RREF FORM FOR A SYSTEM OF EQUATIONS:

0 = nonzero number No solution exists. System is inconsistent.
This takes priority over everything else. Once 0 = nonzero number appears there is no solution possible and all of the other matrix rows are ignored.

0 = 0. The equation corresponding to the given row was dependent on the other equations. Just ignore this completely.

If any row has a nonzero entry in it other than the leading 1 and a number in the final column then there are infinitely many solutions (provided case 1 above does not hold).

Example 6: Systems of equations involving x, y and z produced matrices that reduced to the following reduced row echelon forms (rrefs). State whether or not each system has a solution (is consistent) or does not have a solution (is inconsistent). If it is consistent, state how many solutions there are and what they are.

	RREF of Matrix	Result	Interpretation
a)	$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$	$x = 3$ $y = -2$ $z = 5$	The system is consistent. There is one solution given by (3, -2, 5)
b)	$\begin{bmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$x + 3z = 8$ $y + z = -6$ $0 = 0$	The system is consistent. There are infinitely many solutions given by (8 - 3z, -6 - z, z)

	RREF of Matrix	Result	Interpretation
c)	$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{aligned} x \quad + 3z &= 0 \\ y + z &= 0 \\ 0 &= 1 \end{aligned}$	The system is inconsistent. There are no solutions.
d)	$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{aligned} x &= 4 \\ y &= 3 \\ z &= 7 \\ 0 &= 0 \end{aligned}$	The system is consistent. There is one solution given by (4, 3, 7).
e)	$\begin{bmatrix} 1 & -3 & 4 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{aligned} x - 3y + 4z &= 5 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$	The system is consistent. There are infinitely many solutions given by $(5 + 3y - 4z, y, z)$ (This means that y and z can be given any values, which may be different from each other, and, as long as x is found from $5 + 3y - 4z$, the result is a solution.)
f)	$\begin{bmatrix} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 8 \end{bmatrix}$	$\begin{aligned} x + 3y \quad &= 4 \\ z &= 8 \end{aligned}$	The system is consistent. There are infinitely many solutions given by $(4 - 3y, y, 8)$
g)	$\begin{bmatrix} 1 & 5 & 6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{aligned} x + 5y + 6z &= 0 \\ 0 &= 1 \\ 0 &= 0 \end{aligned}$	The system is inconsistent. There are no solutions.

The matrix corresponding to a system of linear equations can be converted to its equivalent reduced row echelon form (rref) either by hand, using pencil and paper, or by using the TI-89. The procedure used to convert the system to rref by hand is called Gauss-Jordan row reduction. It is a method that incorporates the manipulation of equations illustrated in examples 1 to 3 in a systematic way to produce the reduced row echelon form of the original system. This method is presented in the next section. The section after that shows how to produce the final result by using the TI-89. Before proceeding further, however, some basic matrix terminology should be introduced.

A **matrix** is an array of numbers such as $A = \begin{bmatrix} 3 & 0 & 5 & -8 \\ 2 & -7 & 9 & 12 \\ 1 & 4 & 6 & -5 \end{bmatrix}$

Rows are horizontal: 2 -7 9 12 is the second row, R_2, of the matrix A.

Columns are vertical: $\begin{matrix} 5 \\ 9 \\ 6 \end{matrix}$ is the third column, C_3, of the matrix.

The **order** of matrix A is 3 x 4 (read 3 by 4) because it has 3 rows and 4 columns.
a_{ij} is the entry in the i_{th} row and j_{th} column of the matrix.
Thus $a_{12} = 0$, $a_{21} = 2$, $a_{34} = -5$, $a_{23} = 9$, $a_{32} = 4$, etc.

GAUSS-JORDAN ROW REDUCTION

Consider the following system of equations:

$$\begin{aligned} 3x + 11y &= 5 \\ x + 2y &= 5 \end{aligned}$$

It would not matter if we interchanged them:
(Notice that these are the equations of example 1.)

$$\begin{aligned} x + 2y &= 5 \\ 3x + 11y &= 5 \end{aligned}$$

The solution is not changed if one of them, such as the first one, is multiplied by a nonzero number such as -3:

$$\begin{aligned} -3x - 6y &= -15 \\ 3x + 11y &= 5 \end{aligned}$$

The two equations can be added together to obtain: $5y = -10$

Then the solution to the original system is the same as the solution of this new equation together with any one of the previous equations such as $x + 2y = 5$.

$$\begin{aligned} x + 2y &= 5 \\ 5y &= -10 \end{aligned}$$

That is, $y = -2$ and $x = 9$ (the solution of $x + 2(-2) = 5$) is the solution to the starting equations.

Three operations on a system of equations that do not change the solution are:

Interchange the equations.
Multiply (or divide) an equation by a nonzero number.
Add a multiple of one equation to another equation and replace the other equation by the result.

These correspond exactly to the following three elementary matrix row operations.

Elementary Matrix Row Operations:

Multiply or divide a matrix row by a nonzero number.

Example 7: Multiply row 2 of the following matrix by 1/3 (that is, divide by 3).

$$\begin{bmatrix} 2 & 5 & 6 \\ 3 & 6 & 7 \\ 8 & 4 & 1 \end{bmatrix} \tfrac{1}{3} R_2 \rightarrow R_2. \quad \text{This produces the matrix} \quad \begin{bmatrix} 2 & 5 & 6 \\ 1 & 2 & 7/3 \\ 8 & 4 & 1 \end{bmatrix}$$

Add a multiple of one matrix row to another matrix row.

Example 8: Multiply row 2 of the last matrix above on the right by -2 and add it to row 1. Replace row 1 by the result.

$$\begin{bmatrix} 2 & 5 & 6 \\ 1 & 2 & 7/3 \\ 8 & 4 & 1 \end{bmatrix} -2R_2 + R_1 \rightarrow R_1$$

Since	$-2 R_2$	=	-2	-4	-14/3
and	R_1	=	2	5	6
the sum is	$-2 R_2 + R_1$	=	0	1	4/3

and the resulting matrix is $\begin{bmatrix} 0 & 1 & 4/3 \\ 1 & 2 & 7/3 \\ 8 & 4 & 1 \end{bmatrix}$

Example 9: Multiply row 2 of the last matrix by -8 and add the result to row 3. Replace row 3 by the result.

$$\begin{bmatrix} 0 & 1 & 4/3 \\ 1 & 2 & 7/3 \\ 8 & 4 & 1 \end{bmatrix} -8R_2 + R_3 \rightarrow R_3$$

$-8R_2$	=	-8	-16	-56/3
R_3	=	8	4	1
$-8R_2 + R_3$	=	0	-12	-53/3

The result is $\begin{bmatrix} 0 & 1 & 4/3 \\ 1 & 2 & 7/3 \\ 0 & -12 & -53/3 \end{bmatrix}$

Interchanging rows of a matrix.

Example 10: Interchange rows 1 and 2 of the following matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} R_1 \leftrightarrow R_2 \qquad \text{The result is } \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

The Basic Pivot Operation.

Any nonzero entry of a matrix can be pivoted on. Examples 7, 8 and 9 combined show the result of pivoting on the entry in the second row and first column of the initial matrix in example 7, the number 3. As will be seen later, this would not be the traditional entry to pivot on when solving a system of equations. However, there are business applications which do involve pivoting on entries other than the ones selected in solving systems of equations, and therefore it is wise to know how to pivot on any nonzero entry in a matrix. The basic approach is to make the pivot entry equal to one (by multiplication or division) and then get zeros above and below that one (by adding multiples of the pivot row to the other rows) as exemplified by examples 7, 8 and 9. Usually the operations shown in examples 8 and 9 are performed at the same time. The terminology used in the following procedure description will be explained by the example following it.

Pivot Procedure:

1. Divide the pivot row by the pivot entry.
2. For each nonzero entry above and below the pivot entry in the pivot column:
Multiply the pivot row by the entry with its sign changed.
Then add the result to the row which contains the entry.

Example 11: Pivot on the entry in the third row and second column (the number 3) of

$$A = \begin{bmatrix} -3 & 5 & 7 \\ 2 & 0 & 4 \\ 6 & 3 & -9 \\ 8 & -2 & 11 \end{bmatrix}$$

Solution: Here the pivot entry is $a_{32} = 3$, the pivot row is the third row containing 6, 3 and -9, and the pivot column is the second column containing 5, 0, 3 and -2. The goal is to make $a_{32} = 1$ instead of 3 (by dividing row 3 by 3) and then get zeros above and below that 1 by adding multiples of the new pivot row to the other rows. Notice that in this case the second row already has a zero above the pivot entry 3 and that therefore it does not need to be changed.

Step 1: Divide row 3 by 3. This makes the pivot entry 1.

$$\begin{bmatrix} -3 & 5 & 7 \\ 2 & 0 & 4 \\ 6 & 3 & -9 \\ 8 & -2 & 11 \end{bmatrix} \quad \tfrac{1}{3}R_3 \rightarrow R_3 \quad \text{yields} \quad \begin{bmatrix} -3 & 5 & 7 \\ 2 & 0 & 4 \\ 2 & 1 & -3 \\ 8 & -2 & 11 \end{bmatrix}$$

Step 2: Multiply the new row 3 (the pivot row with the pivot entry now equal to 1) by -5 and add it to row 1. This makes $a_{12} = 0$.
Multiply the new row 3 by 2 and add it to row 4. This makes $a_{42} = 0$.

$$\begin{bmatrix} -3 & 5 & 7 \\ 2 & 0 & 4 \\ 2 & 1 & -3 \\ 8 & -2 & 11 \end{bmatrix} \quad \begin{matrix} -5R_3 + R_1 \rightarrow R_1 \\ \\ 2R_3 + R_4 \rightarrow R_4 \end{matrix} \quad \text{yields} \quad \begin{bmatrix} -13 & 0 & 22 \\ 2 & 0 & 4 \\ 2 & 1 & -3 \\ 12 & 0 & 5 \end{bmatrix}$$

Gauss-Jordan Row Reduction.

Procedure for converting a matrix to reduced row echelon form (rref):

Obtain a nonzero entry in the first row and first column if necessary by interchanging two rows. (Keep in mind the fact that you will divide the first row by this entry and would like to avoid fractions as much as possible.)
Pivot on the nonzero entry in the first row and first column.
Look at the remaining rows. Identify the first column that has a nonzero entry in it that appears below the first row already pivoted on. Get a nonzero entry in the second row in that column. If necessary (or desirable) interchange the second row with a row below it to achieve this purpose.
Pivot on that nonzero entry in the second row.
Repeat this procedure for the remaining rows until every row has a leading 1 or is filled with 0s. In looking for the appropriate nonzero entry to pivot on, NEVER look at the entries in the rows already pivoted on that already have leading 1s.

The best way to understand this procedure is by seeing examples worked out.

Example 12: Use the Gauss-Jordan row reduction procedure to reduce the following matrix to reduced row echelon form (rref): $\begin{bmatrix} 3 & 7 & 5 \\ 2 & 4 & -6 \end{bmatrix}$

Solution 1: $\begin{bmatrix} 3 & 7 & 5 \\ 2 & 4 & -6 \end{bmatrix} \frac{1}{3}R_1 \to R_1$ yields $\begin{bmatrix} 1 & 7/3 & 5/3 \\ 2 & 4 & -6 \end{bmatrix}$

$\begin{bmatrix} 1 & 7/3 & 5/3 \\ 2 & 4 & -6 \end{bmatrix} -2R_1 + R_2 \to R_2$ yields $\begin{bmatrix} 1 & 7/3 & 5/3 \\ 0 & -2/3 & -28/3 \end{bmatrix}$

$\begin{bmatrix} 1 & 7/3 & 5/3 \\ 0 & -2/3 & -28/3 \end{bmatrix} \frac{-3}{2}R_2 \to R_2$ yields $\begin{bmatrix} 1 & 7/3 & 5/3 \\ 0 & 1 & 14 \end{bmatrix}$

$\begin{bmatrix} 1 & 7/3 & 5/3 \\ 0 & 1 & 14 \end{bmatrix} \frac{-7}{3}R_2 + R_1 \to R_1$ yields $\begin{bmatrix} 1 & 0 & -31 \\ 0 & 1 & 14 \end{bmatrix}$

Solution 2: $\begin{bmatrix} 3 & 7 & 5 \\ 2 & 4 & -6 \end{bmatrix} R_1 \leftrightarrow R_2$ yields $\begin{bmatrix} 2 & 4 & -6 \\ 3 & 7 & 5 \end{bmatrix}$

$\begin{bmatrix} 2 & 4 & -6 \\ 3 & 7 & 5 \end{bmatrix} \frac{1}{2}R_1 \to R_1$ yields $\begin{bmatrix} 1 & 2 & -3 \\ 3 & 7 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 7 & 5 \end{bmatrix} -3R_1 + R_2 \to R_2$ yields $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 14 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 14 \end{bmatrix} -2R_2 + R_1 \to R_1$ yields $\begin{bmatrix} 1 & 0 & -31 \\ 0 & 1 & 14 \end{bmatrix}$

While both solutions produce the same result, $\begin{bmatrix} 1 & 0 & -31 \\ 0 & 1 & 14 \end{bmatrix}$, the arithmetic is easier to perform in the second case. Fractions cannot always be avoided, but in some cases (such as this one) they can be avoided or minimized.

Example 13: Find the reduced row echelon form of $\begin{bmatrix} 0 & 2 & -4 & 4 \\ 1 & 3 & -6 & 4 \\ 2 & -5 & 1 & 1 \end{bmatrix}$

Solution: Since 0 appears in the first row and first column and there are nonzero entries in row 2 and row 3, row 1 must be interchanged with one of those two rows. Since row 2 has a 1 as its first entry, this makes interchanging row 1 with row 2 especially desirable.

$$\begin{bmatrix} 0 & 2 & -4 & 4 \\ 1 & 3 & -6 & 4 \\ 2 & -5 & 1 & 1 \end{bmatrix} R_1 \leftrightarrow R_2 \quad \text{yields} \quad \begin{bmatrix} 1 & 3 & -6 & 4 \\ 0 & 2 & -4 & 4 \\ 2 & -5 & 1 & 1 \end{bmatrix}$$

Now all that remains of the first pivot is getting a 0 in the third row and first column.

$$\begin{bmatrix} 1 & 3 & -6 & 4 \\ 0 & 2 & -4 & 4 \\ 2 & -5 & 1 & 1 \end{bmatrix} -2R_1 + R_3 \rightarrow R_3 \quad \text{yields} \quad \begin{bmatrix} 1 & 3 & -6 & 4 \\ 0 & 2 & -4 & 4 \\ 0 & -11 & 13 & -7 \end{bmatrix}$$

Next the entry in the second row and second column is pivoted on.

$$\begin{bmatrix} 1 & 3 & -6 & 4 \\ 0 & 2 & -4 & 4 \\ 0 & -11 & 13 & -7 \end{bmatrix} \tfrac{1}{2} R_2 \rightarrow R_2 \quad \text{yields} \quad \begin{bmatrix} 1 & 3 & -6 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & -11 & 13 & -7 \end{bmatrix}$$

The 0s above and below the 1 in the second row and second column can be obtained at the same time with little risk of error.

$$\begin{bmatrix} 1 & 3 & -6 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & -11 & 13 & -7 \end{bmatrix} \begin{matrix} -3R_2 + R_1 \rightarrow R_1 \\ 11R_2 + R_3 \rightarrow R_3 \end{matrix} \quad \text{yields} \quad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -9 & 15 \end{bmatrix}$$

Next the entry in the third row and third column is pivoted on.

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -9 & 15 \end{bmatrix} \tfrac{-1}{9} R_3 \rightarrow R_3 \quad \text{yields} \quad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -5/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -5/3 \end{bmatrix} 2R_3 + R_2 \rightarrow R_2 \quad \text{yields} \quad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4/3 \\ 0 & 0 & 1 & -5/3 \end{bmatrix}$$

Example 14: Row reduce the given matrix to rref. $\begin{bmatrix} 1 & 3 & -2 & -2 \\ 1 & 3 & -1 & 7 \\ 2 & 6 & -3 & 5 \end{bmatrix}$

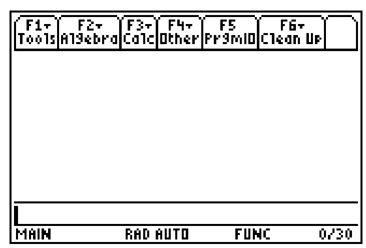
Solution: $\begin{bmatrix} 1 & 3 & -2 & -2 \\ 1 & 3 & -1 & 7 \\ 2 & 6 & -3 & 5 \end{bmatrix} \begin{matrix} -R_1 + R_2 \to R_2 \\ \\ -2R_1 + R_3 \to R_3 \end{matrix}$ yields $\begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 1 & 9 \end{bmatrix}$

Notice that in the second column the only nonzero entry is 3. However, row 2 can onlybe exchanged with a row below it to obtain a nonzero entry in the second row and second column. (Observe that exchanging row 2 with row 1 would undo the previous work done; this is why it is not allowed.) The first column with nonzero entries (ignoring the first row which was already pivoted on and has a leading 1) is now the third column. So the second row and third column provides the pivot entry which is now pivoted on.

$\begin{bmatrix} 1 & 3 & -2 & -2 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 1 & 9 \end{bmatrix} \begin{matrix} 2R_2 + R_1 \to R_1 \\ \\ -R_2 + R_3 \to R_3 \end{matrix}$ yields $\begin{bmatrix} 1 & 3 & 0 & 16 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row 3 is filled with 0s and there is no row below it containing nonzero entries, so the row reduction is now finished.

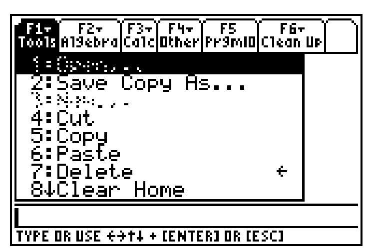
REDUCED ROW ECHELON FORM AND THE TI-89



When the TI-89 calculator is first turned on the screen that appears is called the **home screen**. It looks like the one shown on the right. If some other screen appears, that happened because the calculator turned itself off automatically while that screen had been in use. If that happens, simply press the key labeled HOME on the calculator. The home screen consists of a **toolbar** on top (F1, etc.) followed by a **history window** that should be blank when the calculator is first turned on. If material appears in the space that is blank in the illustration, it can be removed by doing the following:

Press F1. The screen shown to the right appears.

Press 8 (Clear Home)

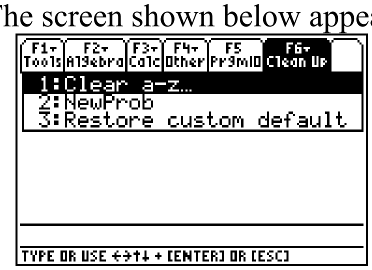


The history window should now be blank. Notice in the display that appeared when F1 (Tools) was pressed options 1 and 3 looked defective. That happened because they corresponded to choices that could not be accessed at the time. When you are dealing with a problem where they are possible options, they will become as legible as the other options that you see. The next to the last line on the display in the Home Screen is called the **entry line** or **command line**. That is where the vertical line called the **cursor** is blinking. If the vertical line is ever accidentally changed to a solid rectangle (overwrite mode), it can be changed back by pressing the yellow 2nd key followed by the ⇐ key which has INS appearing above it in yellow. If anything other than the blinking cursor appears on the command line it can be removed by pressing the CLEAR key. Whenever something new is displayed in this text concerning calculator usage, it is assumed that the history window and command line have been cleared by the key sequence F1, 8, CLEAR.

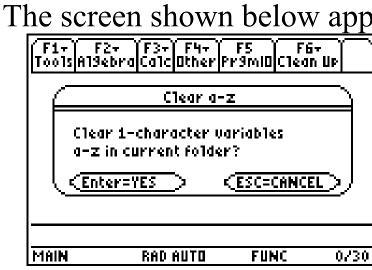
The keys on the calculator have a character (or command) written in white on them. That character or command is accessed by simply pressing that key. In addition, most keys have two other characters or commands appearing above them in some color other than white (yellow, green or purple). For example, the ⇐ key has INS (in yellow) and DEL (in green) above it. To access the INS command you must press the yellow 2nd key and then press ⇐. To access the DEL command you press the green ◆ key and then ⇐. Notice that purple is reserved for letters of the alphabet. Letters of the alphabet (other than x, y, z and t which appear in white) are accessed by pressing the purple alpha key before pressing the appropriate key; for example, to access the letter h the alpha key is pressed and then 8. If it is desired to have the h appear as an upper case character (H), then the key sequence is alpha, then ↑ followed by 8.

Storing a matrix in a TI-89 calculator results in the calculator keeping that matrix in memory until it is erased. This is true whenever any result is stored in a variable. But from a practical point of view, this is usually only done for matrices. Hence, it is best if all variables are cleared after using the calculator with matrices. In the home screen, proceed as follows.

Press F6 (2nd F1)
The screen shown below appears.



Then press ENTER.
The screen shown below appears.



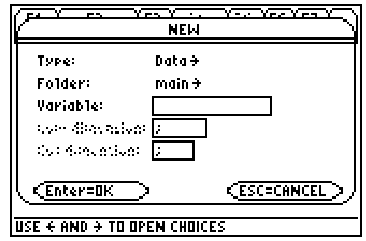
Press ENTER again. The home screen reappears.

The TI-89 will now be used to enter the matrix $A = \begin{bmatrix} 3 & 7 & 5 \\ 2 & 4 & -6 \end{bmatrix}$ which appeared in Example 12. In what appears, the letter *A* will be used as an upper case letter in order to agree with this text and the usual convention of representing a matrix as an upper case letter. However, one key stroke can easily be eliminated by leaving the *A* as a lower case letter (a).

Press APPS

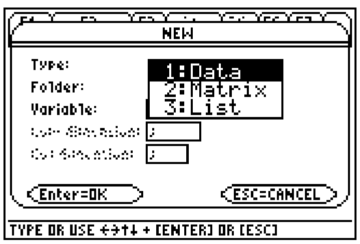
Press 6 (Data/Matrix Ed.)

Press 3 (New)

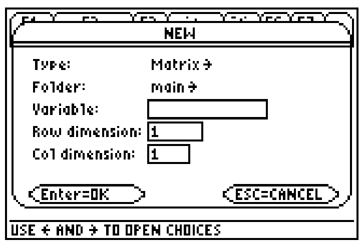


Notice that the word data is blinking. That indicates where the cursor is. The four arrows on the upper right side of the keyboard are used to move the cursor up, down, left and right.

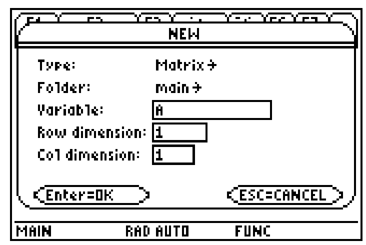
Press right cursor



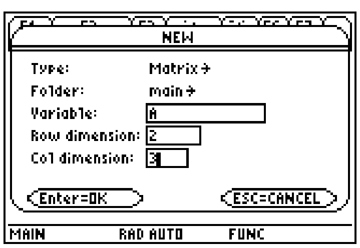
Press 2 (Matrix)



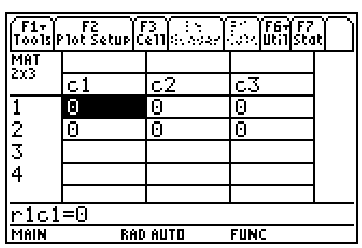
Press down cursor twice so that the cursor blinks in the variable rectangle.
Press alpha, ↑, = (A)



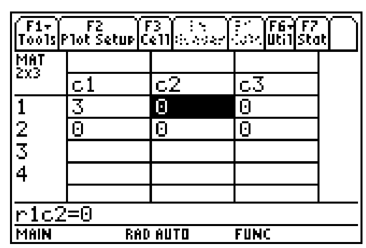
Press down cursor, 2,
down cursor, 3



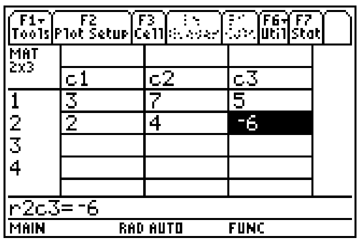
Press ENTER twice



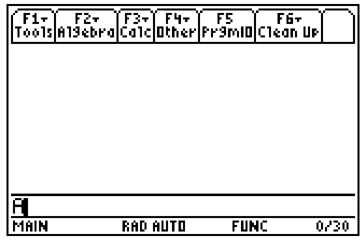
Press 3, ENTER



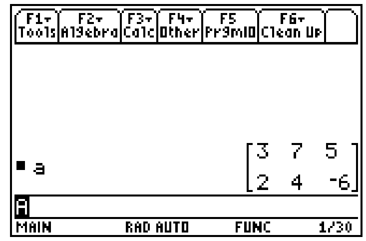
Press 7, ENTER
5, ENTER
2, ENTER
4, ENTER
Grey (-), 6, ENTER
(The cursor movement keys can be used to go back to any entry that needs changing.)



Press HOME
Press alpha, ↑, = (A)



Press ENTER

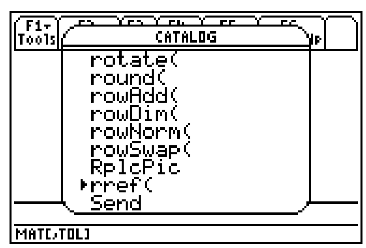


Note that when information is stored in variables the calculator does not distinguish between upper and lower case. So A was entered on the command line but a appeared in the history window.

Apart from going to the home screen, the last command (A, ENTER) was not needed. However, it does represent a useful way to display the final matrix and confirm the fact that the matrix was correctly entered into the calculator.

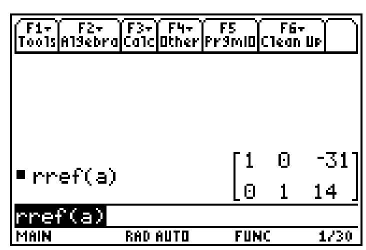
Example 15. Use the TI-89 to find the reduced row echelon form of $A = \begin{bmatrix} 3 & 7 & 5 \\ 2 & 4 & -6 \end{bmatrix}$, the matrix which appeared in Example 12.

Solution: Matrix A was entered into the calculator above. Go to the Home screen and clear the command line. Next press the "catalog" key. Then press 2, which has the letter R in purple above it (when using catalog it is not necessary to press the alpha key before pressing a letter). Scroll down with the down cursor until you reach rref. Position the solid pointer so that it points to rref as shown on the right.



Press enter, alpha, a,), enter.
Your screen should look like the one shown.
The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & -31 \\ 0 & 1 & 14 \end{bmatrix}$$



SOLVING SYSTEMS OF LINEAR EQUATIONS

You are now ready to solve systems of linear equations by carrying out the following steps:

Convert the system of equations to matrix form.
(Make sure all of the variables are lined up correctly. Sometimes a variable is missing and it is very important to enter a 0 in the corresponding matrix position.)
Find the reduced row echelon form of the matrix.
Interpret the result.
If the system is consistent, check the answer.

Example 16: Solve the system $3x + 7y = 5$
$2x + 4y = -6$

Solution: The corresponding matrix, $\begin{bmatrix} 3 & 7 & 5 \\ 2 & 4 & -6 \end{bmatrix}$, is precisely the matrix in examples 12 and 15 and its reduced row echelon form was found in those examples. So the solution is $x = -31$ and $y = 14$. That is, (-31, 14).

Check: $3x + 7y = 5$: $3(-31) + 7(14) = -93 + 98 = 5$ checks.
$2x + 4y = -6$: $2(-31) + 4(14) = -62 + 56 = -6$ checks.

Example 17: Solve the system

$$\begin{aligned} 2y - 4z &= 4 \\ x + 3y - 6z &= 4 \\ 2x - 5y + z &= 1 \end{aligned}$$

Solution: Notice that the sign is included with the number in forming the augmented matrix $\begin{bmatrix} 0 & 2 & -4 & 4 \\ 1 & 3 & -6 & 4 \\ 2 & -5 & 1 & 1 \end{bmatrix}$ whose rref was found in Example 13 to be $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4/3 \\ 0 & 0 & 1 & -5/3 \end{bmatrix}$

You should now make sure you can find the rref by using the TI-89 as well.
(Remember to clear a-z first, if you have not already done so.)
So the solution to the system is $x = -2$, $y = -4/3$ and $z = -5/3$; that is, (-2, -4/3, -5/3).

Check: $2y - 4z = 4$ $\quad 2(\frac{-4}{3}) - 4(\frac{-5}{3}) = \frac{-8}{3} + \frac{20}{3} = \frac{12}{3} = 4$ which checks.

$x + 3y - 6z = 4$ $\quad (-2) + 3(\frac{-4}{3}) - 6(\frac{-5}{3}) = -2 - 4 + 10 = 4$ which checks.

$2x - 5y + z = 1$ $\quad 2(-2) - 5(\frac{-4}{3}) + (\frac{-5}{3}) = -4 + \frac{20}{3} - \frac{5}{3} = -4 + 5 = 1$ checks.

Example 18: Solve the system which appeared on page 2:

$$\begin{aligned} 3x - 5y + 2z &= 11 \\ 2x + 3y - 4z &= 1 \\ 6x - 29y + 20z &= 41 \end{aligned}$$

Solution: The TI-89 shows the rref of $\begin{bmatrix} 3 & -5 & 2 & 11 \\ 2 & 3 & -4 & 1 \\ 6 & -29 & 20 & 41 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -14/19 & 2 \\ 0 & 1 & -16/19 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Note that when you began entering the matrix on the TI-89 only three columns appeared. But when you entered the 2 in the first row the matrix shifted over to reveal the fourth column that was needed.

Thus, $\begin{aligned} x \quad - \tfrac{14}{19} z &= 2 \\ y - \tfrac{16}{19} z &= -1 \\ 0 &= 0 \end{aligned}$ so that $\begin{aligned} x &= 2 + \tfrac{14}{19} z \\ y &= -1 + \tfrac{16}{19} z \\ 0 &= 0 \text{ is ignored} \end{aligned}$

The equations are consistent and there are infinitely many solutions given by

$(2 + \tfrac{14}{19} z, -1 + \tfrac{16}{19} z, z)$ where z is any real number.

check: $3x - 5y + 2z = 11$: $3(2 + \tfrac{14}{19} z) - 5(-1 + \tfrac{16}{19} z) + 2z = 6 + \tfrac{42}{19} z + 5 - \tfrac{80}{19} z + 2z = 11$

$2x + 3y - 4z = 1$: $2(2 + \tfrac{14}{19} z) + 3(-1 + \tfrac{16}{19} z) - 4z = 4 + \tfrac{28}{19} z - 3 + \tfrac{48}{19} z - 4z = 1$

$6x - 29y + 20z = 41$: $6(2 + \tfrac{14}{19} z) - 29(-1 + \tfrac{16}{19} z) + 20z = 12 + \tfrac{84}{19} z + 29 - \tfrac{464}{19} z + 20z = 41$

Example 19: Solve the system:

$$\begin{aligned} 3x - 5y + 2z &= 11 \\ 2x + 3y - 4z &= 1 \\ 6x - 29y + 20z &= 42 \end{aligned}$$

Solution: This is the same system as in example 18 except for the fact that 41 was changed to 42. You can enter the matrix in the calculator from scratch, but it is useful to realize that you can edit a matrix on the TI-89. Assuming you have not cleared the variables yet, press apps, 6:Data/Matrix Editor, 1:Current. The matrix of example 18 now appears. You can use the cursor keys to move to any entry of the matrix and change it. In this case change the 41 to 42. Then press Home, use catalog to enter rref, and thus enter rref (a).

The TI-89 shows the rref of $\begin{bmatrix} 3 & -5 & 2 & 11 \\ 2 & 3 & -4 & 1 \\ 6 & -29 & 20 & 42 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -14/19 & 0 \\ 0 & 1 & -16/19 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The last row indicates the contradiction 0 = 1. Therefore the system is inconsistent and there is no solution.

Example 20: Solve the system $4x + 8y - 16z + 8u = 16$
$x + 2y - 5z + 3u = 7$

Solution: The TI-89 can be used to reduce the system to rref or Gauss-Jordan row reduction can be performed as follows:

$$\begin{bmatrix} 4 & 8 & -16 & 8 & 16 \\ 1 & 2 & -5 & 3 & 7 \end{bmatrix} R_1 \leftrightarrow R_2 \Rightarrow \begin{bmatrix} 1 & 2 & -5 & 3 & 7 \\ 4 & 8 & -16 & 8 & 16 \end{bmatrix} -4R_1 + R_2 \rightarrow R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -5 & 3 & 7 \\ 0 & 0 & 4 & -4 & -12 \end{bmatrix} \tfrac{1}{4}R_2 \rightarrow R_2 \Rightarrow \begin{bmatrix} 1 & 2 & -5 & 3 & 7 \\ 0 & 0 & 1 & -1 & -3 \end{bmatrix} 5R_2 + R_1 \rightarrow R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 & -8 \\ 0 & 0 & 1 & -1 & -3 \end{bmatrix} \Rightarrow \begin{matrix} x + 2y - 2u = -8 \\ z - u = -3 \end{matrix} \Rightarrow \begin{matrix} x = -8 - 2y + 2u \\ z = -3 + u \end{matrix}$$

The system is consistent.
There are infinitely many solutions given by $(-8 - 2y + 2u, y, -3 + u, u)$.

Check: $4x + 8y - 16z + 8u = 16$:
$4(-8 - 2y + 2u) + 8y - 16(-3 + u) + 8u = -32 - 8y + 8u + 8y + 48 - 16u + 8u = 16$
$x + 2y - 5z + 3u = 7$:
$(-8 - 2y + 2u) + 2y - 5(-3 + u) + 3u = - 8 - 2y + 2u + 2y + 15 - 5u + 3u = 7$

Example 21: Solve the system
$2x - 3y = 5$
$-7x + 10y = 4$
$-3x + 4y = 14$

Solution: Gauss-Jordan row reduction or the TI-89 yields the rref $\begin{bmatrix} 1 & 0 & -62 \\ 0 & 1 & -43 \\ 0 & 0 & 0 \end{bmatrix}$.

Thus $x = -62$
$y = -43$
$0 = 0$

The system is consistent with one unique solution given by (-62, -43).

Check: $2x - 3y = 5$: $2(-62) - 3(-43) = -124 + 129 = 5$
$-7x + 10y = 4$: $-7(-62) + 10(-43) = 434 - 430 = 4$
$-3x + 4y = 14$: $-3(-62) + 4(-43) = 186 - 172 = 14$

MATRICES AND MATRIX OPERATIONS

Addition, Subtraction and Scalar Multiplication of Matrices.

In order to add or subtract two matrices, $A = [a_{ij}]$ and $B = [b_{ij}]$, both matrices must have the same n x m order. The matrices are then added or subtracted entry by entry. That is,

$$A + B = [a_{ij} + b_{ij}] \qquad \text{and} \qquad A - B = [a_{ij} - b_{ij}]$$

Multiplication of a matrix by a single number (referred to as a **scalar**), c, is called scalar multiplication and is accomplished by multiplying every entry of the matrix by the scalar. Thus,

$$cA = [ca_{ij}]$$

Example 22: If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 20 & 30 & -40 \\ 50 & -5 & 0 \\ -4 & 1 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 8 \\ -2 & 0 \\ 7 & 1 \end{bmatrix}$

find $A + B$, $A - B$, $-3A$, $4A - 2B$ and $A + C$.

Solution: $A + B = \begin{bmatrix} 1+20 & -2+30 & 3+(-40) \\ -4+50 & 5+(-5) & -6+0 \\ 7+(-4) & 8+1 & 9+(-7) \end{bmatrix} = \begin{bmatrix} 21 & 28 & -37 \\ 46 & 0 & -6 \\ 3 & 9 & 2 \end{bmatrix}$

$$A - B = \begin{bmatrix} 1-20 & -2-30 & 3-(-40) \\ -4-50 & 5-(-5) & -6-0 \\ 7-(-4) & 8-1 & 9-(-7) \end{bmatrix} = \begin{bmatrix} -19 & -32 & 43 \\ -54 & 10 & -6 \\ 11 & 7 & 16 \end{bmatrix}$$

$$-3A = \begin{bmatrix} -3(1) & -3(-2) & -3(3) \\ -3(-4) & -3(5) & -3(-6) \\ -3(7) & -3(8) & -3(9) \end{bmatrix} = \begin{bmatrix} -3 & 6 & -9 \\ 12 & -15 & 18 \\ -21 & -24 & -27 \end{bmatrix}$$

$$4A - 2B = \begin{bmatrix} 4 & -8 & 12 \\ -16 & 20 & -24 \\ 28 & 32 & 36 \end{bmatrix} - \begin{bmatrix} 40 & 60 & -80 \\ 100 & -10 & 0 \\ -8 & 2 & -14 \end{bmatrix} = \begin{bmatrix} -36 & -68 & 92 \\ -116 & 30 & -24 \\ 36 & 30 & 50 \end{bmatrix}$$

A is 3 x 3 and is 3 x 2. Since they have different orders they cannot be added together. So $A + C$ does not exist.

Matrix Multiplication.

Matrix multiplication is not carried out in the same way that addition and subtraction are carried out because that would not be useful in solving problems. Instead, a more complicated procedure is used whose fundamental operation involves multiplying a row times a column.

BASIC MATRIX MULTIPLICATION STEP: ROW TIMES COLUMN.

In order to multiply a row times a column the number of entries in each must be equal.
Multiply the first row entry times the first column entry, then the second row entry times the second column entry, and so on until all entries have been multiplied.
The sum of all the products found is the result of multiplying the row times the column.

Example 23:

Multiply the row 2 3 4 times the column $\begin{matrix} 5 \\ 6 \\ 7 \end{matrix}$

Solution: 2 x 5 = 10   3 x 6 = 18   4 x 7 = 28   10 + 18 + 28 = 56
The result is 56.

MATRIX MULTIPLICATION:

In order to multiply two matrices the number of columns in the first matrix must equal the number of rows in the second matrix.
Each row in the first matrix is multiplied times each column in the second matrix.
The result of multiplying the i^{th} row of the first matrix, R_i, times the j^{th} column of the second matrix, C_j, is put in the i^{th} row and j^{th} column of the result.

Example 24: If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 & 8 \\ 9 & 10 & 11 \end{bmatrix}$ find a) AB b) BA c) A^2 and d) B^2

Solution:

a) The number of columns of A (two) equals the number of rows of B (two), so matrix A can be multiplied times matrix B. Each row of A must be multiplied times each column of B and the result is placed in the corresponding row and column of the answer.

Row 1 times column 1: 2 3 times $\begin{matrix} 6 \\ 9 \end{matrix}$: (2 x 6) + (3 x 9) = 12 + 27 = 39

Row 1 times column 2: (2 x 7) + (3 x 10) = 14 + 30 = 44
Row 1 times column 3: (2 x 8) + (3 x 11) = 16 + 33 = 49
Row 2 times column 1: (4 x 6) + (5 x 9) = 24 + 45 = 69
Row 2 times column 2: (4 x 7) + (5 x 10) = 28 + 50 = 78
Row 2 times column 3: (4 x 8) + (5 x 11) = 32 + 55 = 87

So $AB = \begin{bmatrix} 39 & 44 & 49 \\ 69 & 78 & 87 \end{bmatrix}$

b) The number of columns in B (three) does not equal the number of rows in A (two), so the matrix B cannot be multiplied times the matrix A. BA does not exist.
(Note: Rather than memorize the rule concerning when two matrices can be multiplied, it is possible to just proceed to multiplication in the way indicated. Then observe that when an attempt is made to multiply the first row of B times the first column of A,

6 7 8 times $\begin{matrix} 2 \\ 4 \end{matrix}$ 6 x 2 = 12 7 x 4 = 28 8 x ???

the number of entries are not equal. So BA does not exist.)

c) The definition of 3^2 is 3 times 3. Likewise, the definition of A^2 is A times A. It would be incorrect to square each of the individual entries of A in order to find A^2.

So $A^2 = AA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2(2)+3(4) & 2(3)+3(5) \\ 4(2)+5(4) & 4(3)+5(5) \end{bmatrix} = \begin{bmatrix} 16 & 21 \\ 28 & 37 \end{bmatrix}$

d) $B^2 = BB$. The first matrix in the product is B. The number of columns is 3
The second matrix in the product is also B. The number of rows is 2.
Since the number of columns in the first matrix does not equal the number of rows in the second matrix, the product does not exist even though both matrices in the product have the same order.

Example 25: If $A = \begin{bmatrix} 2 & -3 & 5 \\ 7 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -6 \\ 1 & 8 \\ -2 & 9 \end{bmatrix}$ find AB and BA.

Check your answers by using the TI-89.

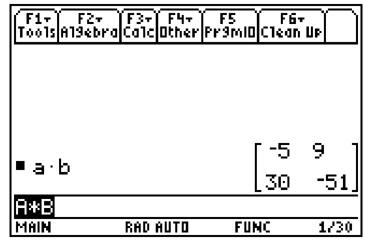
Solution: $AB = \begin{bmatrix} 2 & -3 & 5 \\ 7 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4 & -6 \\ 1 & 8 \\ -2 & 9 \end{bmatrix} = \begin{bmatrix} 8-3-10 & -12-24+45 \\ 28+0+2 & -42+0-9 \end{bmatrix} = \begin{bmatrix} -5 & 9 \\ 30 & -51 \end{bmatrix}$

$BA = \begin{bmatrix} 4 & -6 \\ 1 & 8 \\ -2 & 9 \end{bmatrix}\begin{bmatrix} 2 & -3 & 5 \\ 7 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 8-42 & -12+0 & 20+6 \\ 2+56 & -3+0 & 5-8 \\ -4+63 & 6+0 & -10-9 \end{bmatrix}$

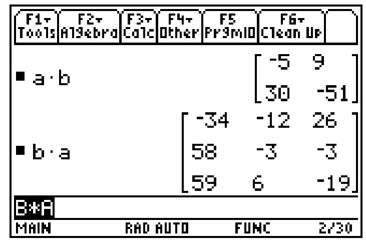
$= \begin{bmatrix} -34 & -12 & 26 \\ 58 & -3 & -3 \\ 59 & 6 & -19 \end{bmatrix}$

Check: Enter matrices *A* and *B* into the calculator.

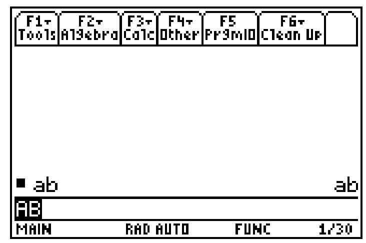
(Remember to clear a-z first.)
Then,while in the home screen:
Press A (alpha, ↑, =), *, B (alpha, ↑, (), ENTER
(* is the multiplication key to the right of 9)
The screen on the right shows what AB equals.



While the A*B is highlighted, Press B (alpha, ↑, ()
(Notice the highlighted A*B disappears and is replaced by B. Whenever the command line is highlighted, typing a new entry erases it. In order to edit the command line either the right cursor or the left cursor would be pressed first.)
Press *, A, ENTER. BA is shown on the screen.

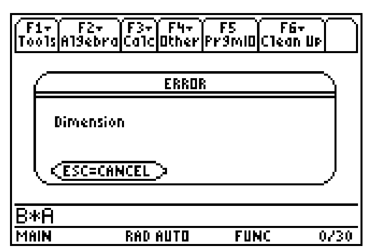


Now clear the windows (F1, 8, CLEAR)
Press A, B, ENTER
Notice that no result is shown. While the calculator understands that 3a means 3 times a, it does not realize that ab means a times b. This is due to the fact that the calculator allows for the possibility that a variable can consist of more than one letter. So the calculator regards ab as the single variable ab rather than a times b. Failure to realize this is the source of some errors. Since the variable ab has not been defined, the calculator does not attempt to display what it is apart from merely repeating the symbols.



IMPORTANT: <u>In general *AB* and *BA* do not produce the same result.</u> The order of multiplication is important.

Remark: If the matrices in Example 24 were entered into the calculator and B*A entered on the command line, then the screen shown on the right would appear. The screen is indicating the fact that the product B*A does not exist because the matrices do not have dimensions (orders) that would make it possible to multiply them.



Example 26: Given $A = \begin{bmatrix} 2 & -3 \\ -5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 8 & -7 \end{bmatrix}$ find: a) AB b) BA c) A^2

d) B^2 e) $A^2 - B^2$ f) $(A - B)(A + B)$ g) $AB - BA$ h) $A^2 - AB$ i) $A(A - B)$

j) In this example $A^2 - B^2$ does not equal $(A - B)(A + B)$ whereas $A^2 - AB = A(A - B)$. Explain why this is true.

Solution: Entering A and B into the TI-89 (after clearing a to z) and performing the operations produces the following results:

a) $A*B = \begin{bmatrix} -16 & 23 \\ 28 & -47 \end{bmatrix}$ b) $B*A = \begin{bmatrix} 3 & -6 \\ 51 & -66 \end{bmatrix}$ c) A^2 (or $A*A$) $= \begin{bmatrix} 19 & -24 \\ -40 & 51 \end{bmatrix}$

d) B^2 $= \begin{bmatrix} 24 & -3 \\ -24 & 57 \end{bmatrix}$ e) A^2-B^2 $= \begin{bmatrix} -5 & -21 \\ -16 & -6 \end{bmatrix}$ f) $(A-B)*(A+B) = \begin{bmatrix} -24 & 8 \\ -39 & 13 \end{bmatrix}$

g) $A*B-B*A = \begin{bmatrix} -19 & 29 \\ -23 & 19 \end{bmatrix}$ h) A^2-$A*B$ $= \begin{bmatrix} 35 & -47 \\ -68 & 98 \end{bmatrix}$ i) $A*(A-B) = \begin{bmatrix} 35 & -47 \\ -68 & 98 \end{bmatrix}$

j) The only usual algebraic operation that does not work for matrices is the commutativity of multiplication; that is, in general AB does not equal BA. As a result, $(A - B)(A + B) = AA + AB - BA - BB = A^2 + AB - BA - B^2$. If these were numbers, the two middle terms would cancel. However, as was seen in part g, they do not cancel for matrices because AB does not equal BA. The reader may also verify the fact that $(A + B)(A - B)$ does not equal the result found in part f for the same reason. Notice that $A(A - B) = AA - AB = A^2 - AB$ does not rely on changing the order of multiplication and hence is always valid as long as the operations are defined.

Inverse of a Matrix.

The remainder of this text is devoted to **square matrices** for which the number of rows equals the number of columns (2 x 2, 3 x 3, 4 x 4, etc.)

Identity Matrix: The n x n matrix I such that, for every other n x n matrix A, $IA = AI = A$.

2 x 2: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 3 x 3: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 4 x 4: $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The identity matrix has 1s on what is called the **main diagonal** and 0s elsewhere.
The identity matrix operates for matrices in the same way as the number 1 operates when it is used in multiplication (1 times 3 equals 3 and 3 times 1 equals 3; $1x = x$).

Example 27: Verify that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 5 \\ -4 & 7 & 1 \\ 0 & 6 & -8 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 5 \\ -4 & 7 & 1 \\ 0 & 6 & -8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 5 \\ -4 & 7 & 1 \\ 0 & 6 & -8 \end{bmatrix}$$

Solution:

You should either verify by hand that the two products equal the result specified (this is easiest) or enter the two matrices into the calculator and find the products to verify it.

Example 28: Given $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, show that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A$ does not equal A.

Solution: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 6 & 8 \end{bmatrix}$ which does not equal A.

Up to this point addition, subtraction, scalar multiplication, and multiplication of matrices have been defined. The identity matrix acts like the number 1 and the matrix filled completely with zeros acts like the number 0. The only thing left to do is to define division. It turns out that this is extremely difficult to do directly and that, in addition, there are some matrices other than the zero matrix that cannot be used as a divisor. So division is approached indirectly. For numbers, dividing by 3 is the same as multiplying by 1/3. That is, 5 ÷ 3 = 5 times (1/3). Also, 1/3 is defined (mathematically) as being the number x such that 3 times x equals x times 3 equals 1; that is, 3 x 1/3 = 1/3 x 3 = 1. Mathematically, this says that 1/3 is the multiplicative inverse of 3. Thus, for real numbers, dividing by a number is the same as multiplying by the multiplicative inverse. This is the approach that works for matrices. The inverse of a matrix is defined and then division by that matrix is equivalent to multiplying by its inverse. Also, recall that 1/3 can be written as 3^{-1}.

Inverse Matrix: The inverse of a square matrix A, if it exists, is denoted by A^{-1} and satisfies

$$AA^{-1} = A^{-1}A = I$$

A matrix A for which A^{-1} does not exist is called a **singular matrix**.

Example 29: Given $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -\frac{1}{2} & 1 \end{bmatrix}$ verify that B is the inverse of A.

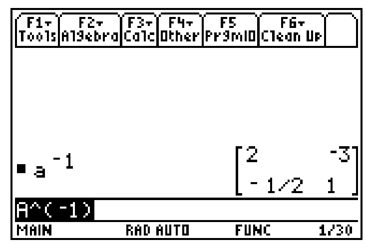
Then use the calculator to find the inverse of A without knowing that B is the inverse.

Solution: To verify that B is the inverse of A means to show that B satisfies the definition given for A^{-1} above; that is, it must be shown that $AB = BA = I$. Since

$$\begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -3 \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

this verifies B is the inverse of A.

Now enter the matrix A in the calculator if this was not already done to perform the multiplications indicated above. (Recall: If the error message “Variable in use so references or changes are not allowed” appears when the variable name A is used in defining the matrix, then the variables from a to z were not cleared by using F6 in the home screen.) With the history window and command line of the home screen cleared, enter the following on the command line:



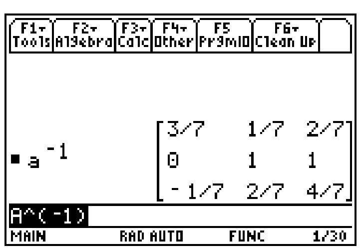
A^(-1)ENTER

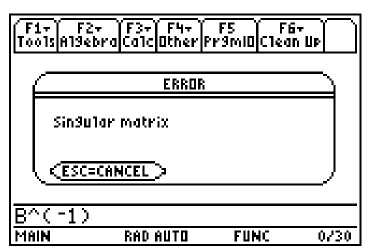
The history window now shows A^{-1}, which is the same as B.

Example 30: Use the calculator to find the inverses of

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -3 \\ 1 & -1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 9 & 5 \\ 1 & 4 & 3 \\ 3 & 14 & 7 \end{bmatrix}$$

Solution: Proceeding as in Example 29 (remember to clear a to z first):





A^{-1} is shown in the screen above. B is singular and has no inverse.

Note that even though the matrix B in Example 30 contains no 0s and, in fact, every entry is positive, the matrix B has no inverse. Division by B is, in effect, impossible. The reason why this is true would take up too much time to develop here, but it is connected with the fact that the row reduced echelon form of B is not the identity matrix while it is for A. Verify this with the TI-89.

Matrix Operations and Equations.

The usual methods for solving equations carry over to matrix equations with two exceptions. The first exception is that division is carried out by multiplying by the inverse. The second exception is that, when multiplying both sides of an equation by a matrix, both sides must be multiplied on the left by the matrix or both sides must be multiplied on the right by the matrix. The reason for this is the fact that AB does not equal BA in general, as has been seen. Thus, if $B = C$ (and A, B and C are all square matrices of the same order), then $AB = AC$ and $BA = CA$ but usually AB does not equal CA and BA does not usually equal AC.

Example 31: Given $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 7 & 9 \end{bmatrix}$ find the 2 x 2 matrix X such that

a) $3X - 4A = B$ b) $AX = B$

Solution:

a) $3X = B + 4A$

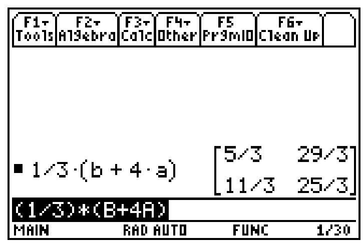
$X = \frac{1}{3}(B + 4A)$

Enter A and B in the calculator.

In the (cleared) home screen enter

(1/3)*(B+4A)ENTER

So $X = \begin{bmatrix} 5/3 & 29/3 \\ 11/3 & 25/3 \end{bmatrix}$



Check: $3X - 4A = 3\begin{bmatrix} 5/3 & 29/3 \\ 11/3 & 25/3 \end{bmatrix} - 4\begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 29 \\ 11 & 25 \end{bmatrix} - \begin{bmatrix} 8 & 24 \\ 4 & 16 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 7 & 9 \end{bmatrix} = B.$

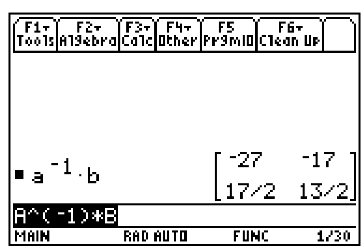
b) X is found by dividing both sides of the equation by A. This is the same as multiplying both sides by A^{-1}. The correct way to do this is as follows:

$A^{-1}AX = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B$

Since A and B are already in the calculator,

A^(-1)*B ENTER

shows that $X = \begin{bmatrix} -27 & -17 \\ 17/2 & 13/2 \end{bmatrix}$



Check: $AX = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -27 & -17 \\ 17/2 & 13/2 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 7 & 9 \end{bmatrix} = B$

It should be noted that there are two incorrect ways of solving this problem

$AXA^{-1} = BA^{-1}$ is algebraically correct but is an incorrect method since AXA^{-1} is not the same as $AA^{-1}X$ (which equals $IX = X$). The reason is that XA^{-1} does not equal $A^{-1}X$. Hence, $AXA^{-1} = BA^{-1}$ just complicates matters.

$A^{-1}AX = BA^{-1}$ is incorrect because if AX is multiplied on the left by A^{-1} then the other side of the equation must be multiplied on the left by A^{-1} and it was multiplied on the right instead. It can be verified on the calculator that

$BA^{-1} = \begin{bmatrix} -17/2 & 14 \\ 19/2 & -12 \end{bmatrix}$ and A times this is $\begin{bmatrix} 40 & -44 \\ 59/2 & -34 \end{bmatrix}$, which is not B.

Rule: The solution of $AX = B$ is $X = A^{-1}B$ and A^{-1} <u>must</u> be on the left.

Good Advice: Always check your answer when possible.

Example 32: Given $A = \begin{bmatrix} 2 & -1 \\ 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$ and $C = \begin{bmatrix} -5 & 6 \\ 2 & 0 \end{bmatrix}$ solve $3A - 6BX = 7C$.

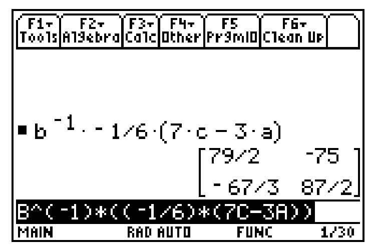
Solution:

$-6BX = 7C - 3A$

$BX = \frac{-1}{6}(7C - 3A)$

We want to divide by B and must do so by multiplying both sides by B^{-1} on the left.

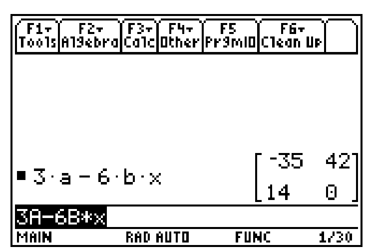
$X = B^{-1}(\frac{-1}{6}(7C - 3A))$



Entering this on the calculator after entering A, B and C produces the answer shown.

That is, the answer is $X = \begin{bmatrix} 79/2 & -75 \\ -67/3 & 87/2 \end{bmatrix}$

The calculator can be used to check this as follows.
With the screen as shown, press the STO key.
Then press x ENTER.
The answer is now stored in the calculator as x.
This can be verified by pressing x ENTER.
Now enter 3A - 6Bx. The screen shown results.
Observe the result is 7C.



(Note: On the command line the multiplication operator * had to be inserted between the B and the x. If it was not, the calculator would not know what the single variable Bx was.)

Alternate Solution showing the individual steps: 3*A* - 6*BX* = 7*C* is

$$\begin{bmatrix} 6 & -3 \\ 24 & 27 \end{bmatrix} - \begin{bmatrix} 18 & 30 \\ 24 & 42 \end{bmatrix} X = \begin{bmatrix} -35 & 42 \\ 14 & 0 \end{bmatrix}$$

$$\Rightarrow - \begin{bmatrix} 18 & 30 \\ 24 & 42 \end{bmatrix} X = \begin{bmatrix} -35 & 42 \\ 14 & 0 \end{bmatrix} - \begin{bmatrix} 6 & -3 \\ 24 & 27 \end{bmatrix} = \begin{bmatrix} -41 & 45 \\ -10 & -27 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 18 & 30 \\ 24 & 42 \end{bmatrix} X = \begin{bmatrix} 41 & -45 \\ 10 & 27 \end{bmatrix}$. Now the inverse of $\begin{bmatrix} 18 & 30 \\ 24 & 42 \end{bmatrix}$ is $\begin{bmatrix} 7/6 & -5/6 \\ -2/3 & 1/2 \end{bmatrix}$

(The inverse is found by calculator)

so that $X = \begin{bmatrix} 7/6 & -5/6 \\ -2/3 & 1/2 \end{bmatrix} \begin{bmatrix} 41 & -45 \\ 10 & 27 \end{bmatrix} = \begin{bmatrix} 79/2 & -75 \\ -67/3 & 87/2 \end{bmatrix}$ (by calculator).

<u>Solving Systems of Equations by Solving Matrix Equations.</u>

Example 33: Solve the system of equations $-2x + 8y = 14$, $x - 3y = 5$ by converting it to a single matrix equation first.

Solution: Notice that $\begin{bmatrix} -2 & 8 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ is the same as $\begin{bmatrix} -2x + 8y \\ x - 3y \end{bmatrix}$.

Hence, the system is equivalent to the single matrix equation

$\begin{bmatrix} -2 & 8 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 5 \end{bmatrix}$ or $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 5 \end{bmatrix}$ whose solution is $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 14 \\ 5 \end{bmatrix}$.

Since, by calculator, $A^{-1} = \begin{bmatrix} 3/2 & 4 \\ 1/2 & 1 \end{bmatrix}$, it follows that

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3/2 & 4 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 14 \\ 5 \end{bmatrix} = \begin{bmatrix} 41 \\ 12 \end{bmatrix}$ so that $x = 41$ and $y = 12$ is the answer.

Check: $-2x + 8y = 14$: $-2(41) + 8(12) = -82 + 96 = 14$
$x - 3y = 5$: $41 - 3(12) = 41 - 36 = 5$

Remarks: Notice that the matrix *A* utilized is always the matrix consisting of the coefficients. This method only works if the number of equations equals the number of variables (so that a square matrix *A* occurs) and the matrix *A* has an inverse.

Example 34: Convert

$$\begin{aligned} 2x \phantom{{}+y} \phantom{{}+z} + u &= 3 \\ -2x \phantom{{}+y} + z + 2u &= -2 \\ -5x \phantom{{}+y} \phantom{{}+z} - 3u &= 4 \\ 7x + y \phantom{{}+z} + 5u &= 1 \end{aligned}$$

to a single matrix equation and solve it. Find the inverse of the coefficient matrix and use it to solve the system of equations.

Solution:

The equation is $A\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 4 \\ 1 \end{bmatrix}$ where $A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ -2 & 0 & 1 & 2 \\ -5 & 0 & 0 & -3 \\ 7 & 1 & 0 & 5 \end{bmatrix}$

Observe that when the data/matrix editor is used to enter A in the TI-89 only three columns can be displayed at a time. At the beginning only c1, c2 and c3 appear but that as soon as c3 is entered the display shifts to show only columns c2, c3 and c4. This happens so that the display can be legible no matter how many columns are needed. The screen will scroll as needed to any column or row desired and the cursor keys can be used to move from one cell to another.

By calculator $A^{-1} = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 4 & 0 & 3 & 1 \\ 16 & 1 & 6 & 0 \\ -5 & 0 & -2 & 0 \end{bmatrix}$ and hence $\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = A^{-1}\begin{bmatrix} 3 \\ -2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 25 \\ 70 \\ -23 \end{bmatrix}$

So the answer is $x = 13$, $y = 25$, $z = 70$ and $u = -23$. That is, (13, 25, 70, -23)

Check:

$2x + u = 3$: $\quad 2(13) + (-23) = 26 - 23 = 3$

$-2x + z + 2u = -2$: $\quad -2(13) + 70 + 2(-23) = -26 + 70 - 46 = -2$

$-5x - 3u = 4$: $\quad -5(13) - 3(-23) = -65 + 69 = 4$

$7x + y + 5u = 1$: $\quad 7(13) + 25 + 5(-23) = 91 + 25 - 115 = 1$

EXERCISES

1. Identify which of the following matrices are not in reduced row echelon form (rref) and state the reason the matrix is not in rref. Also, in each case where the matrix is not in rref, one elementary row operation suffices to get the matrix in rref; state what that row operation is.

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$

d) $\begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ e) $\begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix}$

2 & 3. For the two matrices shown:

a) Identify the order of A.
b) Identify a_{12}, a_{21}, a_{23}, a_{13}, a_{31}, a_{32} and a_{22}.
c) Perform the row operation $R_1 \leftrightarrow R_3$.
d) Perform the operation $-5R_2 \rightarrow R_2$ (on the original matrix A).
e) Perform the operation $2R_3 + R_1 \rightarrow R_1$ (on the original matrix A).
f) Perform the operation $(3/2)R_1 \rightarrow R_1$ (on the original matrix A).
g) Perform the operation $-7R_3 + R_2 \rightarrow R_2$ (on the original matrix A).
h) Pivot on the entry in the third row and first column, a_{31}, of the original matrix A.

2. $\begin{bmatrix} 4 & -2 & 10 & 8 \\ 1 & 5 & -7 & 0 \\ 3 & 6 & 9 & -11 \end{bmatrix}$ 3. $\begin{bmatrix} 5 & 7 & -8 \\ 3 & -4 & -10 \\ 2 & 15 & 1 \\ 6 & 0 & 9 \end{bmatrix}$

4 to 13. Use the Gauss-Jordan row reduction procedure to reduce the following matrices to reduced row echelon form (rref).

4. $\begin{bmatrix} 5 & -15 & 20 \\ 3 & 1 & -18 \end{bmatrix}$ 5. $\begin{bmatrix} 2 & 8 & -10 & 4 \\ 3 & 13 & 7 & -5 \\ 4 & 17 & 5 & 12 \end{bmatrix}$ 6. $\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 & 7 \\ 1 & 0 & 2 & 0 & 5 \\ 0 & 0 & 4 & 0 & 8 \end{bmatrix}$

7. $\begin{bmatrix} 3 & 6 & -12 & 9 \\ 2 & 4 & 2 & -4 \\ 5 & 7 & 4 & -3 \end{bmatrix}$ 8. $\begin{bmatrix} 3 & -6 & 12 \\ 4 & -8 & 16 \end{bmatrix}$ 9. $\begin{bmatrix} 2 & 3 & 7 \\ 6 & 9 & 18 \end{bmatrix}$ 10. $\begin{bmatrix} 1 & -4 & 3 & -2 \\ 2 & -8 & 4 & -4 \\ 4 & -16 & 5 & 3 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 7 & 18 & 6 \\ 1 & 2 & 5 & 2 \\ 2 & 3 & 7 & 4 \end{bmatrix}$ 12. $\begin{bmatrix} 1 & 2 & 9 \\ 2 & 5 & 25 \\ 1 & 1 & 2 \\ 3 & 8 & 41 \end{bmatrix}$ 13. $\begin{bmatrix} 2 & -8 & 2 & 16 & 10 \\ 3 & -12 & 4 & 27 & 22 \end{bmatrix}$

14 to 25. Solve the system of equations shown by using Gauss-Jordan matrix row reduction.

14. $5x - 15y = 20$
$3x + y = -18$
(cf. Exercise 4)

15. $2x + 8y - 10z = 4$
$3x + 13y + 7z = -5$
$4x + 17y + 5z = 12$
(cf. Exercise 5)

16. $y + 3z = 4$
$6z + u = 7$
$x + 2z = 5$
$4z = 8$
(Note: Matrix row reduction must be used.)
(cf. Exercise 6)

17. $3x + 6y - 12z = 9$
$2x + 4y + 2z = -4$
$5x + 7y + 4z = -3$
(cf. Exercise 7)

18. $5x + 3y - 3z = 4$
$x + y - z = 2$
$2x - 6y + 3z = -1$

19. $3x - 6y = 12$
$4x - 8y = 16$
(cf. Exercise 8)

20. $2x + 3y = 7$
$6x + 9y = 18$
(cf. Exercise 9)

21. $x - 4y + 3z = -2$
$2x - 8y + 4z = -4$
$4x - 16y + 5z = 3$
(cf. Exercise 10)

22. $3x + 7y + 18z = 6$
$x + 2y + 5z = 2$
$2x + 3y + 7z = 4$
(cf. Exercise 11)

23. $x + 2y = 9$
$2x + 5y = 25$
$x + y = 2$
$3x + 8y = 41$
(cf. Exercise 12)

24. $2x - 8y + 2z + 16u = 10$
$3x - 12y + 4z + 27u = 22$
(cf. Exercise 13)

25. $x - 2y + 2z + 7u = 3$
$3x - 6y + 7z + 23u = 8$
$x - 2y + 3z + 9u = 3$

26. Given $A = \begin{bmatrix} 2 & 7 \\ 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 9 \\ 3 & 4 \end{bmatrix}$

find: a) $A + B$ b) $B - A$ c) $-8A$ d) $5B + 3A$ e) $2A - 7B$.

27. Given $A = \begin{bmatrix} 3 & -5 & 7 \\ 2 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -6 \\ 8 & 7 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 & -9 \\ 6 & -10 & -3 \end{bmatrix}$ find:

a) $A + C$ b) $A + B$ c) $C + B$ d) $A - C$ e) $B - A$ f) $B + B$ g) $3C$ h) $3C + 2A$ i) $5A - 4C$

28. Given $A = \begin{bmatrix} 5 & -11 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 3 \\ 8 & -9 \end{bmatrix}$ find: a) AB b) BA c) A^2 d) B^2

29. Given $A = \begin{bmatrix} 3 & 5 \\ -7 & 8 \\ 0 & 4 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 & 0 & 10 \\ 11 & 2 & -7 & 8 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -1 \\ 8 & 3 \end{bmatrix}$ find:

a) AB b) BA c) A^2 d) B^2 e) C^2 f) CB g) BC h) AC i) CA
j) In general, what is required for the square of a matrix to exist?
(Hint: What was true of A^2, B^2 and C^2 in this exercise? Why?)

30. Given $A = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ -5 \\ 9 \end{bmatrix}$ find: a) AB b) BA

31. Do Exercise 28 with the TI-89.
(Note: A^2 can be entered in the calculator either as A^2 or as A*A.)

32. Do Exercise 30 with the TI-89.

33. Do Exercise 29 with the TI-89.

34. Given $A = \begin{bmatrix} 3 & 6 & -12 \\ 2 & 4 & 2 \\ 5 & 7 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 7 & 18 \\ 1 & 2 & 5 \\ 2 & 3 & 7 \end{bmatrix}$ use the TI-89 to find:

a) BA b) $AB - B^2$ c) $3A - 5B$ d) $A^3 + 8AB - B^4$ e) the rref of A f) the rref of B

35. Find the inverses of the following matrices and check your answers.

a) $\begin{bmatrix} 14 & -6 \\ 9 & -4 \end{bmatrix}$ b) $\begin{bmatrix} 6 & 8 \\ 3 & 4 \end{bmatrix}$ c) $\begin{bmatrix} 4 & -2 \\ 1 & 3 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 9 & 5 \\ 4 & 7 & 3 \\ 6 & 8 & 3 \end{bmatrix}$ e) $\begin{bmatrix} 3 & 5 & 8 \\ 1 & 7 & 4 \\ 5 & 3 & 12 \end{bmatrix}$

36. Given $A = \begin{bmatrix} 3 & -5 \\ -7 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 16 & 5 \\ 6 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 \\ 5 & 3 \end{bmatrix}$, find the 2 x 2 matrix X such that

a) $X + 4B = C$ b) $5A - 2X = 4B$ c) $BX = A$ d) $CX = B$

e) $BX - 3A = 5C$ f) $2BX + 7A = 4C$

Check your answers.

37. Given $A = \begin{bmatrix} 2 & 9 & 5 \\ 4 & 7 & 3 \\ 6 & 8 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & -1 \\ -3 & 2 & 5 \\ 7 & 1 & 0 \end{bmatrix}$ find the 3 x 3 matrix X such that:

a) $X - A = B$ b) $AX = B$ c) $2X - 3A = 4B$ d) $-2AX = 5B$

Check your answers.

38a) Convert the system $\begin{aligned} 14x - 6y &= 2 \\ 9x - 4y &= 5 \end{aligned}$ to a single matrix equation.

b) Find the inverse of the coefficient matrix.

c) Use the inverse found to solve the system. Check your answer.

39a) Convert the system $\begin{aligned} 3x - 5y + 2z &= -4 \\ -6x + y - 5z &= 2 \\ 8x + 2y + 7z &= 11 \end{aligned}$ into a single matrix equation.

b) Find the inverse of the coefficient matrix.

c) Use the inverse found to solve the system. Check your answer.

ANSWERS TO EXERCISES

1b) The leading nonzero entry in row 2 is 2; $\frac{1}{2}R_2 \to R_2$.

d) -3 appears in the same column as the leading 1 in row 2; $3R_2 + R_1 \to R_1$.

e) The leading 1s do not descend ladder like from left to right; $R_2 \leftrightarrow R_3$.

2a) 3 x 4 b) $a_{12} = -2$, $a_{21} = 1$, $a_{23} = -7$, $a_{13} = 10$, $a_{31} = 3$, $a_{32} = 6$, $a_{22} = 5$

c) $\begin{bmatrix} 3 & 6 & 9 & -11 \\ 1 & 5 & -7 & 0 \\ 4 & -2 & 10 & 8 \end{bmatrix}$ d) $\begin{bmatrix} 4 & -2 & 10 & 8 \\ -5 & -25 & 35 & 0 \\ 3 & 6 & 9 & -11 \end{bmatrix}$ e) $\begin{bmatrix} 10 & 10 & 28 & -14 \\ 1 & 5 & -7 & 0 \\ 3 & 6 & 9 & -11 \end{bmatrix}$

f) $\begin{bmatrix} 6 & -3 & 15 & 12 \\ 1 & 5 & -7 & 0 \\ 3 & 6 & 9 & -11 \end{bmatrix}$ g) $\begin{bmatrix} 4 & -2 & 10 & 8 \\ -20 & -37 & -70 & 77 \\ 3 & 6 & 9 & -11 \end{bmatrix}$ h) $\begin{bmatrix} 0 & -10 & -2 & 68/3 \\ 0 & 3 & -10 & 11/3 \\ 1 & 2 & 3 & -11/3 \end{bmatrix}$

3a) 4 x 3 b) $a_{12} = 7$, $a_{21} = 3$, $a_{23} = -10$, $a_{13} = -8$, $a_{31} = 2$, $a_{32} = 15$, $a_{22} = -4$

c) $\begin{bmatrix} 2 & 15 & 1 \\ 3 & -4 & -10 \\ 5 & 7 & -8 \\ 6 & 0 & 9 \end{bmatrix}$ d) $\begin{bmatrix} 5 & 7 & -8 \\ -15 & 20 & 50 \\ 2 & 15 & 1 \\ 6 & 0 & 9 \end{bmatrix}$ e) $\begin{bmatrix} 9 & 37 & -6 \\ 3 & -4 & -10 \\ 2 & 15 & 1 \\ 6 & 0 & 9 \end{bmatrix}$

f) $\begin{bmatrix} 15/2 & 21/2 & -12 \\ 3 & -4 & -10 \\ 2 & 15 & 1 \\ 6 & 0 & 9 \end{bmatrix}$ g) $\begin{bmatrix} 5 & 7 & -8 \\ -11 & -109 & -17 \\ 2 & 15 & 1 \\ 6 & 0 & 9 \end{bmatrix}$ h) $\begin{bmatrix} 0 & -61/2 & -21/2 \\ 0 & -53/2 & -23/2 \\ 1 & 15/2 & 1/2 \\ 0 & -45 & 6 \end{bmatrix}$

4) $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \end{bmatrix}$ 5) $\begin{bmatrix} 1 & 0 & 0 & 511 \\ 0 & 1 & 0 & -121 \\ 0 & 0 & 1 & 5 \end{bmatrix}$ 6) $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{bmatrix}$ 7) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

8) $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ 9) $\begin{bmatrix} 1 & 3/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 10) $\begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 11) $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

12) $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 13) $\begin{bmatrix} 1 & -4 & 0 & 5 & -2 \\ 0 & 0 & 1 & 3 & 7 \end{bmatrix}$

14) The system is consistent. There is one solution given by (-5, -3).
15) The system is consistent. There is one solution given by (511, -121, 5).
16) The system is consistent. There is one solution given by (1, -2, 2, -5).
17) The system is consistent. There is one solution given by (3, -2, -1).
18) The system is consistent. There is one solution given by (-1, -10/3, -19/3).
19) The system is consistent. There are infinitely many solutions given by $(4 + 2y, y)$.
20) The system is inconsistent. There are no solutions.
21) The system is inconsistent. There are no solutions.
22) The system is consistent. There are infinitely many solutions given by $(2 + z, -3z, z)$.
23) The system is consistent. There is one solution given by (-5, 7).
24) The system is consistent. There are infinitely many solutions given by $(-2 + 4y - 5u, y, 7 - 3u, u)$.
25) The system is inconsistent. There are no solutions.

26a) $\begin{bmatrix} 8 & 16 \\ 8 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}$ c) $\begin{bmatrix} -16 & -56 \\ -40 & -8 \end{bmatrix}$ d) $\begin{bmatrix} 36 & 66 \\ 30 & 23 \end{bmatrix}$ e) $\begin{bmatrix} -38 & -49 \\ -11 & -26 \end{bmatrix}$

27a) $\begin{bmatrix} 5 & -1 & -2 \\ 8 & -10 & -4 \end{bmatrix}$ b) Does not exist c) Does not exist d) $\begin{bmatrix} 1 & -9 & 16 \\ -4 & 10 & 2 \end{bmatrix}$ e) Does not exist f) $\begin{bmatrix} 8 & -12 \\ 16 & 14 \end{bmatrix}$

g) $\begin{bmatrix} 6 & 12 & -27 \\ 18 & -30 & -9 \end{bmatrix}$ h) $\begin{bmatrix} 12 & 2 & -13 \\ 22 & -30 & -11 \end{bmatrix}$ i) $\begin{bmatrix} 7 & -41 & 71 \\ -14 & 40 & 7 \end{bmatrix}$

28a) $\begin{bmatrix} -108 & 114 \\ 0 & -3 \end{bmatrix}$ b) $\begin{bmatrix} -14 & 47 \\ 22 & -97 \end{bmatrix}$ c) $\begin{bmatrix} 3 & -66 \\ 12 & -21 \end{bmatrix}$ d) $\begin{bmatrix} 40 & -39 \\ -104 & 105 \end{bmatrix}$

29a) $\begin{bmatrix} 67 & 7 & -35 & 70 \\ 60 & 23 & -56 & -6 \\ 44 & 8 & -28 & 32 \\ -18 & -5 & 14 & -6 \end{bmatrix}$ b) $\begin{bmatrix} 29 & -8 \\ 27 & 27 \end{bmatrix}$ c) Does not exist d) Does not exist e) $\begin{bmatrix} -4 & -5 \\ 40 & 1 \end{bmatrix}$

(Notice in part (e) that the square of a matrix can have negative entries.)

29f) $\begin{bmatrix} -3 & -4 & 7 & 12 \\ 65 & -2 & -21 & 104 \end{bmatrix}$ g) Does not exist h) $\begin{bmatrix} 46 & 12 \\ 50 & 31 \\ 32 & 12 \\ -14 & -7 \end{bmatrix}$ i) Does not exist

j) Since $A^2 = AA$, and, in order to exist, the number of columns of the first matrix, A, must equal the number of rows of the second matrix, also A, it follows that the number of rows of A must equal the number of columns of A. Such a matrix is called a square matrix.

30a) $\begin{bmatrix} -33 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 8 & -12 \\ -5 & -10 & 15 \\ 9 & 18 & -27 \end{bmatrix}$

31) See #28 32) See #30 33) See #29

34a) $\begin{bmatrix} 113 & 172 & 50 \\ 32 & 49 & 12 \\ 47 & 73 & 10 \end{bmatrix}$ b) $\begin{bmatrix} -61 & -92 & -215 \\ -1 & 2 & 7 \\ 7 & 20 & 53 \end{bmatrix}$ c) $\begin{bmatrix} -6 & -17 & -126 \\ 1 & 2 & -19 \\ 5 & 6 & -23 \end{bmatrix}$

d) $\begin{bmatrix} -9617 & -16,687 & -38,191 \\ -2391 & -4114 & -10,839 \\ -3702 & -6297 & -16,840 \end{bmatrix}$ e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

35a) $\begin{bmatrix} 2 & -3 \\ 9/2 & -7 \end{bmatrix}$ b) Does not exist (singular) c) $\begin{bmatrix} 3/14 & 1/7 \\ -1/14 & 2/7 \end{bmatrix}$ d) $\begin{bmatrix} 3/2 & -13/2 & 4 \\ -3 & 12 & -7 \\ 5 & -19 & 11 \end{bmatrix}$ e) Does not exist (singular)

36a) $\begin{bmatrix} -65 & -20 \\ -19 & -5 \end{bmatrix}$ b) $\begin{bmatrix} -49/2 & -45/2 \\ -59/2 & 6 \end{bmatrix}$ c) $\begin{bmatrix} 41/2 & -15 \\ -65 & 47 \end{bmatrix}$ d) $\begin{bmatrix} -16 & -5 \\ 86/3 & 9 \end{bmatrix}$

e) $\begin{bmatrix} -6 & -165/2 \\ 20 & 261 \end{bmatrix}$ f) $\begin{bmatrix} -395/4 & 75/2 \\ 627/2 & -233/2 \end{bmatrix}$

37a) $\begin{bmatrix} 6 & 9 & 4 \\ 1 & 9 & 8 \\ 13 & 9 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 107/2 & -9 & -34 \\ -97 & 17 & 63 \\ 154 & -27 & -100 \end{bmatrix}$ c) $\begin{bmatrix} 11 & 27/2 & 11/2 \\ 0 & 29/2 & 29/2 \\ 23 & 14 & 9/2 \end{bmatrix}$

d) $\begin{bmatrix} -535/4 & 45/2 & 85 \\ 485/2 & -85/2 & -315/2 \\ -385 & 135/2 & 250 \end{bmatrix}$

38a) $\begin{bmatrix} 14 & -6 \\ 9 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ b) See 35a for A^{-1} c) $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -11 \\ -26 \end{bmatrix}$. So (-11, -26)

39a) $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$ where $A = \begin{bmatrix} 3 & -5 & 2 \\ -6 & 1 & -5 \\ 8 & 2 & 7 \end{bmatrix}$ b) $A^{-1} = \begin{bmatrix} 17 & 39 & 23 \\ 2 & 5 & 3 \\ -20 & -46 & -27 \end{bmatrix}$

c) (263, 35, -309)