1. Rationalization and Indeterminate Forms

Why is rationalization of a denominators so important? Before calculators were used, if we needed to approximate $\frac{4}{\sqrt{3}}$, it would require the division of 4 by 1.7320508. This is an easy division, but it is somewhat time consuming. On the other hand, consider $\frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$, to approximate this equivalent expression we need only multiply 4 by 1.7320508 and divide by 3. Multiplication by a decimal expression is usually faster than division by one. However, if we were using a calculator, it is just as easy to approximate the first expression as the second. Why then do we need to rationalize denominators? We really need it for expressions involving sums and differences of square roots in a denominator when we are performing an evaluation. Recall that zero divided by any non-zero number is 0, while a non-zero number divided by zero is undefined. What happens if we are evaluating an expression and we obtain zero divided by zero? Such an expression is called an *indeterminate* form and it may have meaning. In calculus, such expressions occur regularly and one tool used in evaluating such expressions is rationalization. Consider the expression

Indeterminate forms

$$\frac{h}{2-\sqrt{4+h}}$$

Try to evaluate it with your calculator when h is 0, you will get some kind of error message, depending on the calculator you use. When $h = 0$, both the numerator and denominator of this expression are zero. However, if we rationalize the denominator, we have

$$\frac{h}{2-\sqrt{4+h}} = \frac{h}{(2-\sqrt{4+h})}\frac{(2+\sqrt{4+h})}{(2+\sqrt{4+h})} =$$

$$\frac{h(2+\sqrt{4+h})}{4-(4+h)} = -\frac{h(2+\sqrt{4+h})}{h}$$

This last expression is algebraically equivalent to the original expression when h is not zero. Suppose, *before* we set h equal to zero we continue the simplification. Then we obtain

$$-\frac{h(2+\sqrt{4+h})}{h} = -(2+\sqrt{4+h})$$

If we *now* set $h = 0$, this last expression evaluates to - 4. It is precisely for calculations of the above type that we need to know how to rationalize denominators.

Example 18

Given the expression $\frac{h}{\sqrt{9+h}-3}$.(a) What happens if this expression is evaluated when $h = 0$? (b) Rationalize the denominator of this expression and completely simplify to obtain an equivalent algebraic expression. (c) Evaluate the expression found in (b) when $h = 0$.

Solution

(a) Setting $h = 0$ results in an *indeterminate* form as both the numerator and denominator are 0.

(b)

$$\frac{h}{\sqrt{9+h}-3} = \frac{h}{(\sqrt{9+h}-3)}\frac{(\sqrt{9+h}+3)}{(\sqrt{9+h}+3)} =$$

$$\frac{h(\sqrt{9+h}+3)}{(9+h)-9} = \frac{h(\sqrt{9+h}+3)}{h} = \sqrt{9+h}+3$$

(c) evaluating $\sqrt{9+h}+3$ when $h = 0$ yields 6.

■■■

Sometimes, in problems similar to the last one, we have to rationalize the *numerator* in order to perform the necessary evaluation, as the next example illustrates.

Example 19

Evaluate the expression $\dfrac{\sqrt{9+h}-3}{h}$ by first rationalizing the numerator and simplifying and then set $h = 0$.

Solution

$$\frac{\sqrt{9+h}-3}{h} = \frac{(\sqrt{9+h}-3)}{h}\frac{(\sqrt{9+h}+3)}{(\sqrt{9+h}+3)} = \frac{9+h-9}{h(\sqrt{9+h}+3)} = \frac{h}{h(\sqrt{9+h}+3)} = \frac{1}{(\sqrt{9+h}+3)}$$

Setting $h = 0$, the simplified expression evaluates to 1/6.

■■■

Exercise Set 1

1. Given the expression $\dfrac{h-16}{\sqrt{h}-4}$, simplify the expression by rationalization and after all simplification is completed, evaluate the expression for $h = 16$.

2. Given the expression $\dfrac{\sqrt{4+h}-2}{h}$, simplify the expression by rationalization and after all simplification is completed, evaluate the expression for $h = 0$.

3. Given the expression $\dfrac{2-\sqrt{4+h}}{2h}$, simplify the expression by rationalization and after all simplification is completed, evaluate the expression for $h = 0$.

4. Given the expression $\dfrac{h-9}{\sqrt{h}-3}$, simplify the expression by rationalization and after all simplification is completed, evaluate the expression for $h = 9$.

5. Given the expression $\dfrac{h}{2-\sqrt{4+h}}$, simplify the expression by rationalization and after all simplification is completed, evaluate the expression for $h = 0$.