Behavior of relative entropy in the hydrodynamic scaling limit

by

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Abstract

The dissertation concerns the simple exclusion and Ginzburg-Landau type models on the periodic integer lattice $\mathbb{Z}/n\mathbb{Z}$. The symmetric simple exclusion process and Ginzburg-Landau type models are reversible and we study them under diffusive (parabolic) scaling of time and space. The corresponding conservation laws describing the behavior of the limiting macroscopic densities are the heat equation and a non-linear diffusion equation respectively. They are known to have unique smooth solutions. Asymmetric simple exclusion models are non-reversible and require hyperbolic scaling. It leads to inviscid Burgers type equations. Solutions of such equations are not unique and may develop shocks. Relevant solutions are known to be entropic solutions of these equations.

We study the behavior of the microscopic entropy relative to local Gibbs measures and to the invariant measure. A version of main results obtained for simple exclusion and Ginzburg-Landau type models can be stated as follows:

Let initial configurations be deterministic and possess a macroscopic profile. Then under appropriate scaling of time and space for any positive macroscopic time the specific microscopic entropy converges to the macroscopic entropy as the scaling parameter N goes to infinity.

The logarithmic Sobolev inequality plays an important role in the proofs. Other tools include martingale decomposition, large deviation theorems for local Gibbs states, convex analysis and the theory of multidimensional diffusion processes (for Ginzburg-Landau model).

Contents

Introduction

Systems with a large number of interacting components (molecules, electric charges, etc.) arise naturally in physics. Classical mechanics provides us with the microscopic (at the particle level) equations of evolution of such systems. Since the particles are moving very rapidly it is impossible to follow the motion in the prevailing time scale. Introduction of a suitably scaled version of the original system allows one to describe its evolution by a few slowly changing macroscopic parameters. (As an example of such a parameter we can consider the density of particles at macroscopic level. Its dynamics is described by a partial differential equation which is referred to as the hydrodynamic equation, or conservation law.) The rigorous mathematical justification of such a simplified description is still largely an open problem.

Since about 1970 several examples with stochastic dynamics have been worked out (see, for instance, reviews [3], [17]). These examples are mostly one dimensional and rely to a very large extent on the specifics of the model.

In the last decade entropy methods were developed (see, for example, [7], [20]) which appear to be both flexible and general. The advantage of the entropy method introduced in [7] is that it can be effectively applied to a rather large class of reversible models under very mild conditions on initial data for a microscopic distribution and no a priori assumptions about the regularity of the solutions of the limiting hydrodynamic equation. The method of relative entropy used in [20] works also for non-reversible models (such as, for example, asymmetric simple exclusion) and gives more information about the behavior of microscopic distributions in the scaling limit. The price one needs to pay for this extra information is that a priori assumptions on the smoothness of the solutions of the hydrodynamic equation are required and only local Gibbs states are allowed as initial data. The focus of research conducted in the present work is the behavior of the microscopic distributions with general initial data and without regularity assumptions about the solutions of the hydrodynamic equation.

Chapter 1

Simple exclusion processes

1.1 Preliminaries

Consider indistinguishable particles moving from site to site of the periodic integer lattice $\Lambda_N = \mathbb{Z}/N\mathbb{Z}$ according to the following rules. Each particle waits a random time distributed exponentially with mean 1 then chooses one of the two neighboring sites of Λ_N with probabilities p and q, $p \in [0,1]$, $q = 1-p$, and jumps to the chosen site if that site is not occupied. Otherwise it suppresses the jump and stays at the same place. The state of the system at any time τ is described by a random vector $\eta(\tau) \in \mathbb{X}_N = \{0,1\}^N$. Each component of η is equal to the occupation number of the corresponding site: $\eta_x = 1$ if site x is occupied and $\eta_x = 0$ if it is vacant. Vector $\eta(\tau)$ is called the configuration of particles at time τ. Define the operator $\mathcal{L}_p$ which acts on functions $f : \mathbb{X}_N \to \mathbb{R}$ by the following formula

$$\mathcal{L}_p f(\eta) = \sum_{x\in\Lambda_N} \{p\eta_x(1-\eta_{x+1}) + q\eta_{x+1}(1-\eta_x)\}(f(\eta^{x,x+1}) - f(\eta)), \quad (1.1)$$

where the configuration $\eta^{x,x+1}$ is obtained from η by exchanging x-th and $(x+1)$-th coordinates:

$$\eta_y^{x,x+1} = \begin{cases} \eta_y & \text{if } y \neq x, x+1 \\ \eta_{x+1} & \text{if } y = x, \\ \eta_x & \text{if } y = x+1. \end{cases}$$

This operator is the infinitesimal generator of the simple exclusion process described above. Here p is the probability to jump to the right and q is the probability to jump to the left. The terms $\eta_x(1-\eta_{x+1})$ and $\eta_{x+1}(1-\eta_x)$ describe the exclusion mechanism: no jump is permitted if the chosen site is occupied. Since the notion of a "particle" (occupied site) and a "hole"(empty site) are interchangeable (this is often referred as the particle-hole duality) from now on we assume that $p \in [\frac{1}{2}, 1]$. Noticing that

$$\eta_x\eta_{x+1}(f(\eta^{x,x+1}) - f(\eta)) = 0 \quad (1.2)$$

and

$$(\eta_x + \eta_{x+1})(f(\eta^{x,x+1}) - f(\eta)) = (f(\eta^{x,x+1}) - f(\eta)) \quad (1.3)$$

we can write $\mathcal{L}_p f$ as

$$
\begin{aligned}
\mathcal{L}_p f &= \sum_{x\in\Lambda_N} (p\eta_x + q\eta_{x+1})(f(\eta^{x,x+1}) - f(\eta)) \\
&= \sum_{x\in\Lambda_N} \{q(\eta_x + \eta_{x+1})(f(\eta^{x,x+1}) - f(\eta)) + (p-q)\eta_x(f(\eta^{x,x+1}) - f(\eta))\} \\
&= q \sum_{x\in\Lambda_N} (f(\eta^{x,x+1}) - f(\eta)) + (p-q) \sum_{x\in\Lambda_N} \eta_x(f(\eta^{x,x+1}) - f(\eta)) \\
&= 2q\mathcal{L}_{\frac{1}{2}} f + (p-q)\mathcal{L}_1 f. \qquad (1.4)
\end{aligned}
$$

Another observation is that the number of particles is preserved in time. This implies a natural decomposition of the configuration space $\mathbb{X}_N = \{0,1\}^N$ into "hyperplanes"

$$\mathbb{X}_{N,n} = \big\{\eta \in \mathbb{X}_N \bigm| \sum_{x\in\Lambda_N} \eta_x = n\big\}, \quad n = 0, 1, \ldots, N,$$

each of which consists of all configurations with a fixed number of particles. If the process starts from $\eta(0) \in \mathbb{X}_{N,n}$ then for all τ it stays in $\mathbb{X}_{N,n}$: $\eta(\tau) \in \mathbb{X}_{N,n}$.

Let μ_N^ρ be the product of N Bernoulli measures with density $\rho \in (0,1)$:

$$\mu_N^\rho(\eta) = \prod_{x\in\Lambda_N} \Big(\rho\eta_x + \big(1-\rho\big)\big(1-\eta_x\big)\Big), \qquad \eta \in \mathbb{X}_N. \qquad (1.5)$$

Define

$$\mu_{N,n}(\eta) = \mu_N^\rho\Big(\eta \Bigm| \sum_{x\in\Lambda_N} \eta_x = n\Big) = \binom{N}{n}^{-1}, \qquad \eta \in \mathbb{X}_{N,n}$$

Conditional measures $\mu_{N,n}$ are uniform and do not depend on ρ. The notation "$\mu(\eta),\ \eta \in \mathbb{X}$" below can be understood either as "$\mu_N^\rho(\eta),\ \eta \in \mathbb{X}_N$" or as "$\mu_{N,n}(\eta),\ \eta \in \mathbb{X}_{N,n}$".

For functions $f,\ g$: $\mathbb{X} \to \mathbb{R}$ define their inner product in $L^2(\mathbb{X}, \mu)$ by

$$\langle f, g \rangle_\mu = \sum_{\eta \in \mathbb{X}} f(\eta) g(\eta) \mu(\eta). \tag{1.6}$$

Let $\mathcal{L}_p^*$ be the operator which is formally adjoint to $\mathcal{L}_p$, that is

$$\langle \mathcal{L}_p^* f, g \rangle = \langle f, \mathcal{L}_p g \rangle \qquad \text{for any } f, g : \mathbb{X} \to \mathbb{R}.$$

Lemma 1.1. $\mathcal{L}_p^* = \mathcal{L}_q$.

Proof. In view of (1.4) it is enough to show that $\mathcal{L}_{\frac{1}{2}}^* = \mathcal{L}_{\frac{1}{2}}$ and $\mathcal{L}_1^* = \mathcal{L}_0$. Since $\mu(\eta) = \mu(\eta^{x,x+1})$ we have

$$\begin{aligned}
\langle \mathcal{L}_{\frac{1}{2}} f, g \rangle_\mu &= \frac{1}{2} \sum_{\eta \in \mathbb{X}} \sum_{x \in \Lambda_N} (f(\eta^{x,x+1}) - f(\eta)) g(\eta) \mu(\eta) \\
&= \frac{1}{2} \sum_{\eta \in \mathbb{X}} \sum_{x \in \Lambda_N} f(\eta^{x,x+1}) g(\eta) \mu(\eta) - \frac{1}{2} \sum_{\eta \in \mathbb{X}} \sum_{x \in \Lambda_N} f(\eta) g(\eta) \mu(\eta) \\
&= \frac{1}{2} \sum_{\eta \in \mathbb{X}} \sum_{x \in \Lambda_N} f(\eta)(g(\eta) - g(\eta^{x,x+1})) \mu(\eta) = \langle f, \mathcal{L}_{\frac{1}{2}} g \rangle_\mu
\end{aligned}$$

and also by (1.2)

$$\begin{aligned}
\langle \mathcal{L}_1 f, g \rangle_\mu &= \sum_{\eta \in \mathbb{X}} \sum_{x \in \Lambda_N} \eta_x (1 - \eta_{x+1})(f(\eta^{x,x+1}) - f(\eta)) g(\eta) \mu(\eta) \\
&= \sum_{\eta \in \mathbb{X}} \sum_{x \in \Lambda_N} \eta_x (f(\eta^{x,x+1}) - f(\eta)) g(\eta) \mu(\eta) \\
&= \sum_{\eta \in \mathbb{X}} \sum_{x \in \Lambda_N} \eta_x f(\eta^{x,x+1}) g(\eta) \mu(\eta) - \sum_{\eta \in \mathbb{X}} \Big(\sum_{x \in \Lambda_N} \eta_x \Big) f(\eta) g(\eta) \mu(\eta) \\
&= \sum_{\eta \in \mathbb{X}} \sum_{x \in \Lambda_N} \eta_{x+1} f(\eta) g(\eta^{x,x+1}) \mu(\eta) - \sum_{\eta \in \mathbb{X}} \Big(\sum_{x \in \Lambda_N} \eta_{x+1} \Big) f(\eta) g(\eta) \mu(\eta) \\
&= \sum_{\eta \in \mathbb{X}} \sum_{x \in \Lambda_N} \eta_{x+1} (g(\eta^{x,x+1}) - g(\eta)) f(\eta) \mu(\eta) = \langle f, \mathcal{L}_0 g \rangle_\mu.
\end{aligned}$$

□

Lemma 1.2. *For any function $f:\ \mathbb{X} \to \mathbb{R}$ we have that $\langle \mathcal{L}_p^* f, 1\rangle_\mu = 0$, i.e. μ is an invariant measure for the simple exclusion process with the generator $\mathcal{L}_p$.*

Proof. Lemma 1.1 and the definition of $\mathcal{L}_q$ imply that

$$\langle \mathcal{L}_p^* f, 1\rangle_\mu = \langle f, \mathcal{L}_q 1\rangle_\mu = 0.$$

□

For a function $f:\ \mathbb{X} \to \mathbb{R}$ define the Dirichlét form $D(f)$ by

$$D(f) = -\langle \mathcal{L}_p f, f\rangle_\mu.$$

Let us point out that $D(f)$ does not depend on p. Indeed, noticing that $\mathcal{L}^*_{\frac{1}{2}} = \mathcal{L}_{\frac{1}{2}}$ and $(\mathcal{L}_p - \mathcal{L}_q)^* = -(\mathcal{L}_p - \mathcal{L}_q)$ we can decompose $\mathcal{L}_p$ into symmetric and antisymmetric parts as follows:

$$\begin{aligned}\mathcal{L}_p &= \frac{1}{2}(\mathcal{L}_p + \mathcal{L}_p^*) + \frac{1}{2}(\mathcal{L}_p - \mathcal{L}_p^*) = \frac{1}{2}(\mathcal{L}_p + \mathcal{L}_q) + \frac{1}{2}(\mathcal{L}_p - \mathcal{L}_q)\\ &= \mathcal{L}_{\frac{1}{2}} + \frac{1}{2}(\mathcal{L}_p - \mathcal{L}_q),\end{aligned}$$

Denoting $\frac{1}{2}(\mathcal{L}_p - \mathcal{L}_q)$ by $\mathcal{L}_a$ we obtain

$$\begin{aligned}\langle \mathcal{L}_p f, f\rangle_\mu &= \langle (\mathcal{L}_{\frac{1}{2}} + \mathcal{L}_a) f, f\rangle_\mu\\ &= \langle \mathcal{L}_{\frac{1}{2}} f, f\rangle_\mu = \frac{1}{2}\sum_{\eta\in\mathbb{X}}\sum_{x\in\Lambda_N} (f(\eta^{x,x+1}) - f(\eta))f(\eta)\mu(\eta)\\ &= \frac{1}{4}\sum_{\eta\in\mathbb{X}}\sum_{x\in\Lambda_N} \Big(2f(\eta)f(\eta^{x,x+1}) - f^2(\eta) - f^2(\eta^{x,x+1})\Big)\mu(\eta)\\ &= -\frac{1}{4}\sum_{\eta\in\mathbb{X}}\sum_{x\in\Lambda_N} \Big(f(\eta^{x,x+1}) - f(\eta)\Big)^2\mu(\eta).\end{aligned}$$

This computation shows that the Dirichlét form (being a quadratic form) depends only on the symmetric part of the generator and therefore is the same for all simple exclusion processes introduced above:

$$D(f) = \frac{1}{4} \sum_{\eta \in \mathbb{X}} \sum_{x \in \Lambda_N} \Big(f(\eta^{x,x+1}) - f(\eta) \Big)^2 \mu(\eta) \geq 0. \tag{1.7}$$

Let ν be an arbitrary probability measure on the configuration space $\mathbb{X}$ and $f(\eta) \overset{\text{def}}{=} \frac{\nu(\eta)}{\mu(\eta)}$. The relative entropy of measure ν with respect to the reference measure μ is given by

$$H(f) \equiv H(\nu \,|\, \mu) = \sum_{\eta \in \mathbb{X}} f(\eta) \log f(\eta) \mu(\eta)$$

At all points where $f(\eta) = 0$ we agree to set $f(\eta) \log f(\eta)$ to zero.

Lemma 1.3. *Let* $f^\tau(\eta): \; [0,T] \times \mathbb{X} \to [0,+\infty)$ *be a solution of the forward equation*

$$\frac{df^\tau}{d\tau} = \mathcal{L}_p^* f^\tau$$

with initial data $f_0 \geq 0$ *which satisfies the normalization condition* $\sum_{\eta \in \mathbb{X}} f_0(\eta)\mu(\eta) = 1$. *Then for any* $\tau \in [0,T]$

$$\frac{dH(f^\tau)}{d\tau} \leq -2D(\sqrt{f^\tau}\,).$$

Proof. A direct computation yields:

$$\begin{aligned}
\frac{dH(f^\tau)}{d\tau} &= \sum_{\eta \in \mathbb{X}} \left(\frac{df^\tau}{d\tau} \right) (\log f^\tau)\, \mu(\eta) + \sum_{\eta \in \mathbb{X}} \frac{df^\tau}{d\tau} \mu(\eta) \\
&= \sum_{\eta \in \mathbb{X}} \left(\mathcal{L}_p^* f^\tau \right) (\log f^\tau)\, \mu(\eta) + \sum_{\eta \in \mathbb{X}} \left(\mathcal{L}_p^* f^\tau \right) \mu(\eta) \\
&= \langle \mathcal{L}_p^* f^\tau, \log f^\tau \rangle_\mu = \langle f^\tau, \mathcal{L}_p \log f^\tau \rangle_\mu = 2\langle f^\tau, \mathcal{L}_p \log \sqrt{f^\tau} \rangle_\mu \\
&\leq 2\langle f^\tau, \frac{1}{\sqrt{f^\tau}} \mathcal{L}_p \sqrt{f^\tau} \rangle_\mu = 2\langle \sqrt{f^\tau}, \mathcal{L}_p \sqrt{f^\tau(\eta)} \rangle_\mu = -2D(\sqrt{f^\tau(\eta)}\,)
\end{aligned}$$

The only inequality in this chain is an application of Jensen's inequality and is a consequence of the concavity of the logarithm (see also Lemma 5.3 in [8]). We have:

$$\begin{aligned}\mathcal{L}_p \log \sqrt{f^\tau}\,(\eta) &= \sum_{x\in\Lambda_N} (p\eta_x + q\eta_{x+1})\Big(\log\sqrt{f^\tau(\eta^{x,x+1})} - \log\sqrt{f^\tau(\eta)}\Big)\\ &\le \frac{1}{\sqrt{f^\tau(\eta)}} \sum_{x\in\Lambda_N} (p\eta_x + q\eta_{x+1})\Big(\sqrt{f^\tau(\eta^{x,x+1})} - \sqrt{f^\tau(\eta)}\Big)\\ &= \frac{1}{\sqrt{f^\tau(\eta)}}\mathcal{L}_p\sqrt{f^\tau}\,(\eta).\end{aligned}$$

□

1.2 Symmetric simple exclusion $(p = q)$

1.2.1 Notation and main results

We study the symmetric simple exclusion process under the diffusive scaling of time and space. For every $N \in \mathbb{N}$ the mapping $(\tau, x) \mapsto (\tau/N^2, x/N)$ takes a point x of the periodic lattice Λ_N to the point $\frac{x}{N}$ of the circle $S = \mathbb{R}/\mathbb{Z}$. It also changes the waiting time rate from 1 to $1/N^2$. We denote macroscopic time by t, $t = \tau/N^2$. Below we consider the scaled process on the lattice $\frac{x}{N} \in S,\ x \in \Lambda_N$.

The following definition will be used throughout the paper:

Definition 1.1. *Let $\eta^{(N)}$, $N = 1, 2, \dots,$ be a set of random configurations and $\nu^{(N)}$ be the corresponding probability measures on $\mathbb{X}_N$: $\nu^{(N)}(A) =$* Prob $(\eta^{(N)} \in A)$ *for any $A \subset \mathbb{X}_N$. We shall say that this set possesses an asymptotic profile $m \in L^1(S)$ as $N \to \infty$ and denote this by $\eta^{(N)} \sim m$ if for* *any test function $J \in C(S)$ and any $\delta > 0$*

$$\lim_{N\to\infty} \nu^{(N)} \left\{ \eta \in \mathbb{X}_N \, : \, \left| \frac{1}{N} \sum_{x\in\Lambda_N} J\left(\frac{x}{N}\right) \eta_x - \int\limits_S J(\theta) m(\theta)\, d\theta \right| > \delta \right\} = 0.$$

Remark. In the case when configurations $\eta^{(N)}$ are deterministic the relation $\eta^{(N)} \sim m_0$ simply means that for any $J \in C(S)$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x\in\Lambda_N} J\left(\frac{x}{N}\right) \eta_x^{(N)} = \int\limits_S J(\theta) m(\theta)\, d\theta.$$

Let ν_N^t be a probability measure on the configuration space $\mathbb{X}_N$ which describes the distribution of $\eta^{(N)}(t)$ given that at $t = 0$ the process started from deterministic configuration $\eta^{(N)}(0)$. Due to the conservation law it is supported on $\mathbb{X}_{N,n_N}$, where $n_N = \sum\limits_{x\in\Lambda_N} \eta_x^{(N)}(0)$. Restricting ν_N^t to $\mathbb{X}_{N,n_N}$ we obtain a measure ν_{N,n_N}^t. Its density f_{N,n_N}^t with respect to the invariant measure μ_{N,n_N} is found from the solution of the following problem:

$$\frac{\partial f}{\partial t} = N^2 \mathcal{L}_{\frac{1}{2}} f$$
$$f|_{t=0} = \delta_{\eta^{(N)}(0)}$$

This problem has a unique solution f_N^t which is supported on $\mathbb{X}_{N,n_N}$ and $f_{N,n_N}^t = \binom{N}{n_N} f_N^t|_{\mathbb{X}_{N,n_N}}$.

Let $\mathcal{M}_1(S)$ be the space of measures on S with total variation bounded by 1. Being endowed with weak-$*$ topology, the space $\mathcal{M}_1(S)$ is a separable metrizable complete topological space. Given a stochastic process $\eta^{(N)}(t)$, $t \in [0, T]$, such that $\eta^{(N)}(0) \sim m_0$ as for some $m_0 \in L^1(S)$, we can construct a process $\chi^{(N)}(t, \cdot),\ t \in [0, T]$, with values in $\mathcal{M}_1(S)$ by setting

$$\chi^{(N)}(t) = \frac{1}{N} \sum_{x\in\Lambda_N} \eta_x^{(N)}(t) \delta_{\frac{x}{N}}.$$

The random measure $\chi^{(N)}(t)$ is called the empirical measure. It describes the distribution of particles on S at time t and at the scaling level N. The trajectories of $\chi^{(N)}(t)$ belong to the Skorokhod space $D([0,T],\mathcal{M}_1(S))$ of measure-valued paths with discontinuities of the first kind only. Consider the corresponding measures Q_N on $D([0,T],\mathcal{M}_1(S))$. If $\eta^{(N)}(0) \sim m_0$ then (see, for example, [16]) the family Q_N is tight and all its limit points are supported on $C([0,T],\mathcal{M}_1(S))$. Moreover if Q is any limit point then for Q-almost all $\chi \in D([0,T],\mathcal{M}_1(S))$ the measure $\chi(t,d\theta)$ has a density $m(t,\theta)$ with respect to Lebesgue measure on S for all $0 \le t \le T$. This density satisfies the heat equation on S with the initial condition m_0. Uniqueness theorem for the heat equation implies that Q has to be the only limit point, it is supported on a single trajectory $m(t,\theta)d\theta$ and $Q^{(N)} \Rightarrow \delta_{m(\cdot,\theta)d\theta}$ as $N \to \infty$. We state a version of these results for future reference:

Theorem 1.1. *Let $\eta^{(N)}(0) \sim m_0$ for some $m_0 \in L^1(S)$ and $\eta^{(N)}(t)$ be the symmetric simple exclusion process with initial data $\eta^{(N)}(0)$. Let m be the solution of the heat equation*

$$\frac{\partial m}{\partial t} = \frac{1}{2}\frac{\partial^2 m}{\partial \theta^2}, \quad (t,\theta) \in (0,+\infty) \times S, \tag{1.8}$$

which satisfies the condition $m\big|_{t=0} = m_0$. Then $\eta^{(N)}(t) \sim m(t,\cdot)$ uniformly in t.

Let $m(t,\theta)$ be a solution of (1.8) for some $t > 0$. We define a family of local Gibbs measures $\gamma_N^{m_t}$ on $\mathbb{X}_N$ by

$$\gamma_N^{m_t}(\eta) = \prod_{x \in \Lambda_N} \Big(m\big(t, \tfrac{x}{N}\big)\eta_x + \big(1 - m(t, \tfrac{x}{N})\big)\big(1 - \eta_x\big) \Big), \quad \eta \in \mathbb{X}_N. \tag{1.9}$$

In other words $\gamma_N^{m_t}$ is a product measure on $\{0,1\}^N$ whose i-th marginal is a Bernoulli measure with probability of "success" given by the macroscopic particle density $m(t, \frac{x}{N})$: $\gamma_N^{m_t}(\eta_x = 1) = m(t, \frac{x}{N})$. Since for each N the process "lives" on $\mathbb{X}_{N,n_N}$ we are interested in the conditionals

$$\gamma_{N,n_N}^{m_t}(\cdot) = \gamma_N^{m_t}(\cdot \mid \sum_{x\in\Lambda_N} \eta_x^{(N)} = n_N). \tag{1.10}$$

Denote by $g_{N,n_N}^{m_t}$ the density of $\gamma_{N,n_N}^{m_t}$ with respect to the invariant measure μ_{N,n_N}, i. e.

$$g_{N,n_N}^{m_t}(\eta) = \frac{\gamma_{N,n_N}^{m_t}(\eta)}{\mu_{N,n_N}(\eta)}, \qquad \eta \in \mathbb{X}_{N,n_N}. \tag{1.11}$$

The main result of this section is the following theorem.

Theorem 1.2. *Let $\eta^{(N)}(0)$, $N = 1, 2, \ldots$, be a sequence of deterministic configurations such that $\eta^{(N)}(0) \sim m_0$ for some $m_0 \in L^1(S)$ and $n_N = \sum_{x\in\Lambda_N} \eta_x^{(N)}(0)$. Then for any $t > 0$ the specific relative entropy*

$$\frac{1}{N} H(\nu_{N,n_N}^t \mid \gamma_{N,n_N}^{m_t}) = \frac{1}{N} \sum_{\eta\in\mathbb{X}_N} f_{N,n_N}^t(\eta) \log \frac{f_{N,n_N}^t(\eta)}{g_{N,n_N}^{m_t}(\eta)}\, \mu_{N,n_N}(\eta)$$

approaches zero as $N \to \infty$.

Define

$$h(y) \stackrel{\text{def}}{=} \begin{cases} y\log y + (1-y)\log(1-y), & y \in (0,1); \\ 0, & y = 0, 1. \end{cases} \tag{1.12}$$

Clearly $h \in C([0,1])$ and convex. Taking into account Lemma 1.4 below we immediately obtain

Corollary. *Under assumptions of Theorem 1.2*

$$\lim_{N\to\infty} \frac{1}{N} H(\nu_{N,n_N}^t \mid \mu_{N,n_N}) = \int_S h(m(t,\theta))\, d\theta - h(\bar{m}). \tag{1.13}$$

This corollary states that for any positive macroscopic time the specific microscopic entropy converges to the macroscopic entropy defined by the right hand side of (1.13).

1.2.2 Proof of Theorem 1.2

At first notice that

$$\frac{1}{N}H(\nu^t_{N,n_N}\,|\,\gamma^{m_t}_{N,n_N}) = \frac{1}{N}H(\nu^t_{N,n_N}\,|\,\mu_{N,n_N}) - \frac{1}{N}E_{\nu^t_{N,n_N}}\log g^{m_t}_{N,n_N}(\eta).$$

Since the relative entropy is always non-negative it is enough to show that the upper limit as $N\to\infty$ of the right hand side is non-positive. The statement of the Theorem 1.2 will follow from the next two lemmas.

Lemma 1.4. *Under assumptions of Theorem 1.2*

$$\lim_{N\to\infty}\left(-\frac{1}{N}E_{\nu^t_{N,n_N}}\log g^{m_t}_{N,n_N}(\eta)\right) = -\int\limits_S h(m(t,\theta))\,d\theta + h(\int\limits_S m(t,\theta)\,d\theta),$$

where h is given by (1.12).

Notice that the "total mass" $\bar{m} = \int\limits_S m(t,\theta)\,d\theta$ is a conserved quantity. This can be easily verified using (1.8).

Lemma 1.5. *Under assumptions of Theorem 1.2*

$$\overline{\lim_{N\to\infty}}\,\frac{1}{N}H(\nu^t_{N,n_N}\,|\,\mu_{N,n_N}) \le \int\limits_S h(m(t,\theta))\,d\theta - h(\int\limits_S m(t,\theta)\,d\theta), \qquad (1.14)$$

Proof of Lemma 1.4. This lemma is a consequence of the existence of the hydrodynamic scaling limit and the choice of reference measures $\gamma^{m_t}_{N,n_N}$.

Since

$$g^{m_t}_{N,n_N}(\eta) = \frac{\gamma^{m_t}_N(\eta)}{\mu_{N,n_N}(\eta)\sum\limits_{\eta\in\mathbb{X}_{N,n_N}}\gamma^{m_t}_N(\eta)}, \qquad \eta\in\mathbb{X}_{N,n_N},$$

we have that

$$-\frac{1}{N}\log g^{m_t}_{N,n_N}(\eta) = -\frac{1}{N}\log\binom{N}{n_N} - \frac{1}{N}\log\gamma^{m_t}_N(\eta) + \frac{1}{N}\log\sum_{\eta\in\mathbb{X}_{N,n_N}}\gamma^{m_t}_N(\eta). \quad (1.15)$$

The first term is easily computed to be $h(\bar{m})$ using Stirling's formula and the assumption $\eta^{(N)}(0)\sim m_0$. The latter implies, in particular, that

$$\lim_{N\to\infty}\frac{n_N}{N} = \int_S m_0(\theta)\,d\theta = \bar{m}.$$

Hence

$$\begin{aligned}-\lim_{N\to\infty}\frac{1}{N}\log\binom{N}{n_N} &= -\lim_{N\to\infty}\frac{1}{N}\log\frac{N^N}{n_N^{n_N}(N-n_N)^{N-n_N}}\\ &= \lim_{N\to\infty}\left(\frac{n_N}{N}\log\frac{n_N}{N} + \left(1-\frac{n_N}{N}\right)\log\left(1-\frac{n_N}{N}\right)\right) = h(\bar{m}) \quad (1.16)\end{aligned}$$

Next we show that

$$\lim_{N\to\infty}\frac{1}{N}E_{\nu^t_{N,n_N}}\log\gamma^{m_t}_N(\eta) = \int_S h(m(t,\theta)).$$

At first we rewrite $\log\gamma^{m_t}_N(\eta)$ as

$$\begin{aligned}&\log\prod_{x\in\Lambda_N}\left(m(t,\tfrac{x}{N})\right)^{\eta_x}\left(1-m(t,\tfrac{x}{N})\right)^{1-\eta_x}\\ &\quad= \sum_{x\in\Lambda_N}\left[\eta_x\log m(t,\tfrac{x}{N}) + (1-\eta_x)\log(1-m(t,\tfrac{x}{N}))\right]\\ &\quad= \sum_{x\in\Lambda_N}J(t,\tfrac{x}{N})\eta_x + \sum_{x\in\Lambda_N}\log(1-m(t,\tfrac{x}{N})),\end{aligned}$$

where

$$J(t,\theta) = \log m(t,\theta) - \log(1-m(t,\theta)). \quad (1.17)$$

The function J is continuous on $(0, +\infty) \times S$ since m is continuous and bounded away from 0 and 1 for each fixed $t > 0$. The last assertion follows from the inequalities $0 \leq m_0(\theta) \leq 1$ and the strong maximum principle for solutions of the heat equation. By Theorem 1.1 we have that $\eta^{(N)}(t) \sim m(t, \cdot)$ and therefore

$$\lim_{N\to\infty} E_{\nu^t_{N,n_N}} \left| \frac{1}{N} \sum_{x\in\Lambda_N} J(t, \frac{x}{N})\eta_x - \int_S J(t,\theta)m(t,\theta)\, d\theta \right| = 0. \tag{1.18}$$

As a consequence we obtain

$$\lim_{N\to\infty} \frac{1}{N} E_{\nu^t_{N,n_N}} \log g^{m_t}_{N,n_N}(\eta) = \int_S J(t,\theta)m(t,\theta)\, d\theta + \int_S \log(1 - m(t,\theta))\, d\theta \tag{1.19}$$

$$= \int_S h(m(t,\theta))\, d\theta. \tag{1.20}$$

Finally by Lemma A.1 we find that

$$\lim_{N\to\infty} \frac{1}{N} \log \sum_{\eta\in\mathbb{X}_{N,n_N}} \gamma^{m_t}_N(\eta) = 0.$$

This together with (1.16) and (1.19) completes the proof of Lemma 1.4. □

Proof of Lemma 1.5. As a preliminary step we prove a simple proposition which we call (following [21]) the martingale decomposition. Throughout the paper we agree to set $0 \log 0$ to zero.

Proposition 1.1. *Let $(\Omega, \mathcal{G}_0, P)$ be a probability space and $\mathcal{G}_0 \supset \mathcal{G}_1 \supset \ldots \supset \mathcal{G}_k$ be a decreasing sequence of σ-algebras. For a non-negative random variable X_0, $EX_0 < \infty$, define $X_i = E\left(X_0 | \, \mathcal{G}_i\right)$, $i = 0, 1, \ldots, k$. Then*

$$E\, X_0 \log X_0 = \sum_{i=1}^{k} E\, E\left(X_{i-1} \log \frac{X_{i-1}}{X_i} \Big| \, \mathcal{G}_i \right) + E\, X_k \log X_k.$$

Proof. We have

$$E\,X_0 \log X_0 = \sum_{i=1}^{k} E\left(X_{i-1}\log X_{i-1} - X_i \log X_i\right) + E\,X_k \log X_k.$$

Taking conditional expectations we find that

$$\begin{aligned}\sum_{i=1}^{k} E\,(X_{i-1}\log X_{i-1} \; & -X_i \log X_i) \\ &= \sum_{i=1}^{k} E\,E\,\left(X_{i-1}\log X_{i-1} - X_i \log X_i \,|\, \mathcal{G}_i\right) \\ &= \sum_{i=1}^{k} E\,E\,\left(X_{i-1}\log X_{i-1} - X_{i-1} \log X_i \,|\, \mathcal{G}_i\right) \\ &= \sum_{i=1}^{k} E\,E\,\left(X_{i-1}\log \frac{X_{i-1}}{X_i} \,|\, \mathcal{G}_i\right).\end{aligned}$$

□

We apply the above proposition in the following context. Let $\Omega = \mathbb{X}_{N,n_N}$ and $P = \mu_{N,n_N}$. Fix any $k \in \mathbb{N}$ and divide the circle S into k equal "arcs" $\Delta_i,\ i = 1, 2, \ldots, k$, of length $\varepsilon = \frac{1}{k}$. Each "arc"will contain $[\varepsilon N]$ or $[\varepsilon N] + 1$ sites. To simplify the notation we shall disregard this difference and shall assume that each "arc"has the same number of sites $l = \varepsilon N \in \mathbb{N}$. Define

$$\bar{n}_i = \sum_{j=(i-1)l+1}^{il} \eta_j, \quad i = 1, 2, \ldots, k, \text{ (number of particles in the } i\text{-th "arc")}$$

$$\begin{aligned}\mathcal{G}_i = \{ & \sigma\text{-algebra generated by coordinate functions } \eta_{il+1}, \ldots, \eta_N \\ & \text{and by } \bar{n}_1, \ldots, \bar{n}_i \text{ subject to the restriction} \\ & \textstyle\sum_{j=1}^{i} \bar{n}_i + \sum_{j=il+1}^{N} \eta_x = n_N \}\ , \quad i = 0, \ldots, k.\end{aligned}$$

Fix an arbitrary positive t and consider f^t_{N,n_N} as a random variable on $(\mathbb{X}_{N,n_N}, \mu_{N,n_N})$. We have $E_{\mu_{N,n_N}} f^t_{N,n_N} = 1$. Let $f^t_i = E_{\mu_{N,n_N}}\left(f^t_{N,n_N} \,|\, \mathcal{G}_i\right)$.

Whenever it does not lead to a confusion the dependence of $\mathcal{G}_i$ and f_i^t on N and n_N will not be reflected in the notation.

By Proposition 1.1 we have

$$\begin{aligned}\frac{1}{N}H(\nu^t_{N,n_N}\,|\,\mu_{N,n_N}) &= \frac{1}{N}E_{\mu_{N,n_N}} f^t_{N,n_N}\log f^t_{N,n_N}\\ &= \frac{1}{N}\sum_{i=1}^{k} E_{\mu_{N,n_N}} E_{\mu_{N,n_N}}\left(f^t_{i-1}\log\frac{f^t_{i-1}}{f^t_i}\,\Big|\,\mathcal{G}_i\right)\\ &+\frac{1}{N}E_{\mu_{N,n_N}} f^t_k \log f^t_k. \end{aligned} \tag{1.21}$$

We are going to show that for sufficiently large k the sum of the first k terms on the right hand side of (1.21) can be made arbitrarily small as $N\to\infty$ by making use of the logarithmic Sobolev inequality. The entropy bound (1.14) then comes from the last term of (1.21).

Step 1. At first we state the logarithmic Sobolev inequality. It was proven in [21] (see also [10] for a simpler proof).

Theorem 1.3. *Let $\mu_{l,n}$ be the uniform measure on $\mathbb{X}_{l,n}=\{\eta\in\{0,1\}^l : \sum_{x=1}^{l}\eta_x=n\}$, i.e. $\mu_{l,n}(\eta)=\binom{l}{n}^{-1}$ for any $\eta\in\mathbb{X}_{l,n}$. For a function f on $\mathbb{X}_{l,n}$ define*

$$\bar{D}_{l,n}(f)=E_{\mu_{l,n}}\sum_{x=1}^{l-1}[f(\eta^{x,x+1})-f(\eta)]^2.$$

Then there is a constant C independent of n and l such that the inequality

$$E_{\mu_{l,n}} f\log f\le Cl^2\bar{D}_{l,n}(\sqrt{f}).$$

holds for any f which is non-negative and satisfies the normalization condition $E_{\mu_{l,n}}f=1$.

Let $\mu_{N,n_N}(\cdot\,|\,\mathcal{G}_i)=\mu_i$. It is a measure on the first i "arcs". Denote by $\bar{\mu}_i$ its marginal on the i-th "arc". It is easy to see that $\bar{\mu}_i$ is a uniform measure.

The function f_i^t depends only on n_N, the last $N-il$ coordinates, and averages $\bar{n}_1, \ldots, \bar{n}_i$. Therefore it can be treated as a constant relative to $\bar{\mu}_i$. The ratio $\bar{f}_i = f_{i-1}^t / f_i^t$ is $\bar{\mu}_i$-measurable and satisfies the condition $E_{\bar{\mu}_i} \bar{f}_i = 1$. By the logarithmic Sobolev inequality

$$E_{\bar{\mu}_i}\left(\bar{f}_i \log \bar{f}_i\right) \leq Cl^2 E_{\bar{\mu}_i} \sum_{x=(i-1)l+1}^{il-1} \left(\sqrt{\bar{f}_i(\eta^{x,x+1})} - \sqrt{\bar{f}_i(\eta)}\right)^2$$

$$\leq Cl^2 \, \frac{1}{f_i^t} \, E_{\bar{\mu}_i} \sum_{x=(i-1)l+1}^{il-1} \left(\sqrt{f_{i-1}^t(\eta^{x,x+1})} - \sqrt{f_{i-1}^t(\eta)}\right)^2. \quad (1.22)$$

Notice that by the definition of f_{i-1} and the Hölder inequality, for each x between $(i-1)l+1$ and $il-1$ we have that

$$\left(\sqrt{f_{i-1}^t(\eta^{x,x+1})} - \sqrt{f_{i-1}^t(\eta)}\right)^2$$

$$= \left(\sqrt{E_{\mu_{i-1}} f_{N,n_N}^t(\eta^{x,x+1})} - \sqrt{E_{\mu_{i-1}} f_{N,n_N}^t(\eta)}\right)^2$$

$$= E_{\mu_{i-1}} \left(\sqrt{f_{N,n_N}^t(\eta^{x,x+1})} - \sqrt{f_{N,n_N}^t(\eta)}\right)^2$$

$$+2\left(E_{\mu_{i-1}} \sqrt{f_{N,n_N}^t(\eta^{x,x+1}) f_{N,n_N}^t(\eta)} - \sqrt{E_{\mu_{i-1}} f_{N,n_N}^t(\eta^{x,x+1})} \sqrt{E_{\mu_{i-1}} f_{N,n_N}^t(\eta)}\right)$$

$$\leq E_{\mu_{i-1}} \left(\sqrt{f_{N,n_N}^t(\eta^{x,x+1})} - \sqrt{f_{N,n_N}^t(\eta)}\right)^2. \quad (1.23)$$

Here we also used the fact that even though μ_{i-1} depends on the last $N-(i-1)l$ coordinates of η it is invariant under the transformation which takes η to $\eta^{x,x+1}$ for all $x \geq (i-1)l+1$. Substituting (1.23) into (1.22), taking summation over i, and averaging with respect to μ_{N,y_N} we obtain

$$\sum_{i=1}^{k} E_{\mu_{N,n_N}} \, E_{\mu_{N,n_N}} \left(f_{i-1}^t \log \frac{f_{i-1}^t}{f_i^t} \mid \mathcal{G}_i\right)$$

$$\leq Cl^2 E_{\mu_{N,n_N}} \sum_{x=1}^{N-1} \left[\sqrt{f_{N,n_N}^t(\eta^{x,x+1})} - \sqrt{f_{N,n_N}^t(\eta)}\right]^2$$

$$\leq 4C\varepsilon^2 D_{N,n_N}(\sqrt{f_{N,n_N}^t}). \quad (1.24)$$

Step 2. We now estimate the Dirichlét form. Since by Lemma 1.3

$$\frac{d}{dt}H(\nu^t_{N,n_N} \mid \mu_{N,n_N}) \le -2D_{N,n_N}(\sqrt{f^t_{N,n_N}}) < 0, \tag{1.25}$$

for all k which satisfy $1/k \le t$ we find

$$\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} D_{N,n_N}(\sqrt{f^\tau_{N,n_N}})\,d\tau \le \frac{1}{2\varepsilon}H(\nu^0_{N,n_N} \mid \mu_{N,n_N}), \quad \varepsilon = \frac{1}{k}.$$

This implies that for each N there exists $t_N \in (t-\varepsilon, t)$ such that

$$D_{N,n_N}(\sqrt{f^{t_N}_{N,n_N}}) \le \frac{1}{2\varepsilon}H(\nu^0_{N,n_N} \mid \mu_{N,n_N}) \le \frac{1}{2\varepsilon}\log\binom{N}{n_N}. \tag{1.26}$$

The last inequality follows from the obvious Proposition 1.2 stated below.

Proposition 1.2. *Let $\mathcal{P}$ be the space of probability measures on $\Omega = \{1, 2, \ldots, M\}$. Then for any P, $Q \in \mathcal{P}$ with $P(i) = p_i$, $Q(i) = q_i$, $i = 1, \ldots, M$,*

$$H(P \mid Q) = \sum_{i=1}^{M} p_i \log\frac{p_i}{q_i} \le \max_{1\le i\le M}\log\frac{1}{q_i}.$$

In our case $q_i \equiv q = \binom{N}{n_N}^{-1}$. From (1.21), (1.23), (1.25), and (1.26) we conclude that

$$\begin{aligned}\frac{1}{N}H(\nu^t_{N,n_N} \mid \mu_{N,n_N}) &\le \frac{1}{N}H(\nu^{t_N}_{N,n_N} \mid \mu_{N,n_N})\\ &\le \frac{2C\varepsilon}{N}\log\binom{N}{n_N} + \frac{1}{N}E_{\mu_{N,n_N}} f^{t_N}_k \log f^{t_N}_k.\end{aligned} \tag{1.27}$$

In the proof of Lemma 1.4 we have already shown that

$$\lim_{N\to\infty}\frac{1}{N}\log\binom{N}{n_N} = -h(\bar{m}).$$

From the definition of h (see (1.12) it is easy to see that $0 \le -h(y) \le \log 2$ for all $y \in [0, 1]$. Now we only need to obtain a bound on the last term of (1.27).

Step 3. The function $f_k^{t_N}$ depends only on $\bar{n}_1, \dots, \bar{n}_k$. Let $\mu^(k)$ be the joint distribution of $\bar{n}_1, \dots, \bar{n}_k$ under μ_{N,n_N}. Taking summation over the set $\{\eta \in \mathbb{X}_{N,n_N} \ : \ \sum_{j=(i-1)l+1}^{il} \eta_j = \bar{n}_i, \ i = 1, \dots, k\}$ we see that

$$\begin{aligned}\frac{1}{N} E_{\mu_{N,n_N}} f_k^{t_N} \log f_k^{t_N} &= \frac{1}{N} E_{\mu^(k)} f_k^{t_N} \log f_k^{t_N} \\ &= \frac{1}{N} \sum_{\bar{n}_1 + \dots + \bar{n}_k = n_N} (f_k^{t_N} \mu^(k))(\bar{n}_1, \dots, \bar{n}_k) \log \big(f_k^{t_N} \mu^(k)(\bar{n}_1, \dots, \bar{n}_k) \big) \\ &- \frac{1}{N} E_{\mu_{N,n_N}} f_k^{t_N} \log \mu^(k)(\bar{n}_1, \dots, \bar{n}_k) \end{aligned} \tag{1.28}$$

Let $\mu^(k)$ be the joint distribution of $\bar{n}_1, \dots, \bar{n}_k$ under μ_{N,n_N}. The first term in the right hand side of (1.28) is non-positive and we concentrate on the second one. We have

$$\mu^(k)(\bar{n}_1, \dots, \bar{n}_k) = \binom{N}{n_N}^{-1} \prod_{i=1}^{k} \binom{l}{\bar{n}_i}$$

and therefore

$$\frac{1}{N} \log \mu^(k)(\bar{n}_1, \dots, \bar{n}_k) = -\frac{1}{N} \log \binom{N}{n_N} + \frac{1}{k} \sum_{i=1}^{k} \frac{1}{l} \log \binom{l}{\bar{n}_i},$$

Returning to (1.28) we can compute

$$\begin{aligned} \overline{\lim_{N\to\infty}} \Big(-\frac{1}{N} E_{\mu_{N,n_N}} f_k^{t_N} \log \mu^(k)(\bar{n}_1, \dots, \bar{n}_k) \Big) &= \\ = - h(\bar{m}) + \frac{1}{k} \sum_{i=1}^{k} \overline{\lim_{N\to\infty}} \Big(-\frac{1}{N\varepsilon} E_{\nu_{N,n_N}^{t_N}} \log \binom{N\varepsilon}{\bar{n}_i} \Big) \quad & (1.29) \\ = - h(\bar{m}) + \frac{1}{k} \sum_{i=1}^{k} h(k \int_{\Delta_i} m(t_\varepsilon, \theta) \, d\theta) & \end{aligned}$$

for some $t_\varepsilon \in [t - \varepsilon, t]$. The last equality is the consequence of the Theorem 1.1. Indeed, Theorem 1.1 implies that for any $\delta > 0$

$$\lim_{N\to\infty} \nu_{N,n_N}^{t} (\eta \in \mathbb{X}_{N,n_N} \ : \ \Big| \frac{\bar{n}_i}{\varepsilon N} - k \int_{\Delta_i} m(t, \theta) \, d\theta \Big| > \delta) = 0$$

uniformly in t. From here we proceed similarly to (1.16) understanding the convergence as convergence in probability relative to ν^t_{N,n_N} and using continuity and boundedness of h.

Conclusion. Combining (1.27) and (1.29) we see that

$$\limsup_{N\to\infty} \frac{1}{N} H(\nu^t_{N,n_N} \mid \mu_{N,n_N}) \leq \frac{1}{k} C \log 2 - h(\bar{m}) + \frac{1}{k} \sum_{i=1}^{k} h(k \int_{\Delta_i} m(t_\varepsilon, \theta)\, d\theta)$$

with $t_\varepsilon \in [t - \frac{1}{k}, t]$. Notice that $|\Delta_i| = \frac{1}{k}$ and $k \int_{\Delta_i} m(t_\varepsilon, \theta)\, d\theta = m(t_\varepsilon, \theta_i)$ for some $\theta_i \in \Delta_i$. Finally we let k go to infinity and obtain the statement of the lemma. □

1.3 Asymmetric simple exclusion ($p \neq q$)

1.3.1 Main results

The appropriate scaling for asymmetric simple exclusion processes is hyperbolic. It is given by the mapping $(\tau, x) \mapsto (\tau/N, x/N)$ which shrinks the distance between sites of Λ_N by N and speeds up the time by the same factor therefore changing the expected waiting time between jumps from 1 to $1/N$. The macroscopic time t is now related to the microscopic time τ by the equation $t = \tau/N$. We consider the scaled process on the lattice $\frac{x}{N} \in S,\ x \in \Lambda_N$ which starts from deterministic initial conditions $\eta^{(N)}(0)$. We adopt the notations introduced in Section 1.2 for the symmetric simple exclusion model. The following result is well-known for asymmetric simple exclusion processes on the circle (and also on the line) (see, for example, [12], [19], and [14]) and it will serve us as the starting point.

Theorem 1.4. *Let $\eta^{(N)}(0) \sim m_0$ for some measurable m_0, $0 \le m_0 \le 1$, and $\eta^{(N)}(t)$ be the asymmetric simple exclusion process with initial data $\eta^{(N)}(0)$. Let m be the entropic solution of the inviscid Burgers' equation*

$$\frac{\partial m}{\partial t} + (p-q)\frac{\partial m(1-m)}{\partial \theta} = 0, \quad (t,\theta) \in (0,+\infty) \times S, \tag{1.30}$$

which satisfies the condition $m\big|_{t=0} = m_0$. Then $\eta^{(N)}(t) \sim m(t,\cdot)$ uniformly in t.

Define a family of local Gibbs measures $\gamma_N^{m_t}$ on $\mathbb{X}_N$ by (1.9) where m now is the entropic solution of (1.30). Introduce $\gamma_{N,n_N}^{m_t}$ and $g_{N,n_N}^{m_t}$ by relations (1.10) and (1.11) respectively.

The main result of this section is

Theorem 1.5. *Let $\eta^{(N)}(0)$, $N = 1, 2, \ldots$, be a sequence of deterministic configurations such that $\eta^{(N)}(0) \sim m_0$ for some measurable m_0: $S \to [\delta, 1-\delta]$, where $0 < \delta < \frac{1}{2}$. Denote $\sum_{x \in \Lambda_N} \eta_x^{(N)}(0)$ by n_N. Then for any $t > 0$ the specific relative entropy*

$$\frac{1}{N} H(\nu_{N,n_N}^t \,|\, \gamma_{N,n_N}^{m_t}) = \frac{1}{N} \sum_{\eta \in E^{(N)}} f_{N,n_N}^t(\eta) \log \frac{f_{N,n_N}^t(\eta)}{g_{N,n_N}^{m_t}(\eta)} \, \mu_{N,n_N}(\eta)$$

approaches zero as $N \to \infty$.

The proof of Theorem 1.5 is given in the next three subsections.

This theorem can be slightly modified to produce the following version:

Theorem 1.6. *Let $\eta^{(N)}(0)$, $N = 1, 2, \ldots$, be a sequence of deterministic configurations such that $\eta^{(N)}(0) \sim m_0$ for some measurable m_0, $0 \le m_0 \le 1$. Then for any $t > 0$*

$$\lim_{N \to \infty} \frac{1}{N} H(\nu_{N,n_N}^t \,|\, \mu_{N,n_N}) = \int_S h(m(t,\theta))\, d\theta - h(\bar{m}), \tag{1.31}$$

where h is defined by (1.12)

This result states that for any positive macroscopic time the specific microscopic entropy converges to the macroscopic entropy defined by the right hand side of (1.31).

We would like to point out that under assumptions of Theorem 1.5 the relation (1.31) is an immediate consequence of Theorem 1.5 and Lemma 1.6 below. We need a separate argument to include functions m_0 which are not bounded away from 0 or 1. To this end we notice that the upper bound on the relative entropy can be proved (see Lemma 1.7) under relaxed assumptions of Theorem 1.6. Therefore we need only a lower bound which is not hard to obtain. We prove this in Subsection 1.3.5.

1.3.2 Proof of Theorem 1.5

The general framework of the proof coincides with the one for the symmetric simple exclusion process. But due to the difference in the scaling, details of this proof are very different from the symmetric case.

We have

$$\frac{1}{N} H(\nu^t_{N,n_N} \mid \gamma^{m_t}_{N,n_N}) = \frac{1}{N} H(\nu^t_{N,n_N} \mid \mu_{N,n_N}) - \frac{1}{N} E_{\nu^t_{N,n_N}} \log g^{m_t}_{N,n_N}.$$

Since the relative entropy is always non-negative the following two lemmas imply the statement of Theorem 1.5.

Lemma 1.6. *Under assumptions of Theorem 1.5*

$$\lim_{N\to\infty} \frac{1}{N} E_{\nu^t_{N,n_N}} \log g^{m_t}_{N,n_N} = \int_S h(m(t,\theta))\, d\theta - h(\bar{m}),$$

where h is defined by (1.12) and

$$\bar{m} = \int_S m(t,\theta)\, d\theta.$$

Notice that the "total mass" $\bar{m}$ is a conserved quantity. This is easily verified using the hydrodynamic equation.

Proof of Lemma 1.6. The proof just repeats the proof of the corresponding lemma for the symmetric case. The only difference is that the function $J(t,\theta) = \log m(t,\theta) - \log(1-m(t,\theta))$ (see (1.17)) might not be continuous. But it is bounded in θ for any $t > 0$ since we assumed that m_0 was bounded away from 0 and 1. Moreover $J(t,\cdot)$ is of bounded variation. This follows from the fact that $m(t,\cdot)$ is of bounded variation on S (see [15], p. 267, discussion below Theorem 16.1). We claim that the convergence (1.18) also takes place for any function $J = J(\theta)$ of bounded variation on S. At first observe that if J is the indicator function of an interval, then (1.18) holds. Clearly we can take a finite linear combination of indicator functions of intervals. Any function of bounded variation can be decomposed into the sum of a continuous function and a jump function. We need only to prove (1.18) for the jump function. Recall (see [9]) that for each jump function f on S there is a finite or countable set of points $\theta_1,\ \theta_2,\ldots$ and corresponding numbers $a_1,\ a_2,\ldots$ and $b_1,\ b_2,\ldots$ such that

$$\sum_n (|a_n| + |b_n|) < \infty$$

and

$$f(\theta) = \sum_{\theta_n \le \theta} a_n + \sum_{\theta_n < \theta} b_n.$$

For any $\varepsilon > 0$ and all $n \geq n_\varepsilon$ we have that

$$\sum_{n \geq n_\varepsilon} (|a_n| + |b_n|) < \varepsilon.$$

Define

$$f_\varepsilon(\theta) = \sum_{n \geq n_\varepsilon \,:\, \theta_n \leq \theta} a_n + \sum_{n \geq n_\varepsilon \,:\, \theta_n < \theta} b_n;$$
$$\tilde{f}(\theta) = f(\theta) - f_\varepsilon(\theta).$$

Then

$$\left| \frac{1}{N} \sum_{x \in \Lambda_N} f\left(\frac{x}{N}\right) \eta_x - \int_0^1 f(\theta) m(t,\theta)\, d\theta \right| \leq$$
$$\leq \left| \frac{1}{N} \sum_{x \in \Lambda_N} f_\varepsilon\left(\frac{x}{N}\right) \eta_x - \int_0^1 f_\varepsilon(\theta) m(t,\theta)\, d\theta \right| +$$
$$+ \left| \frac{1}{N} \sum_{x \in \Lambda_N} \tilde{f}\left(\frac{x}{N}\right) \eta_x - \int_0^1 \tilde{f}(\theta) m(t,\theta)\, d\theta \right| \leq$$
$$\leq 2\varepsilon + \left| \frac{1}{N} \sum_{x \in \Lambda_N} \tilde{f}\left(\frac{x}{N}\right) \eta_x - \int_0^1 \tilde{f}(\theta) m(t,\theta)\, d\theta \right|.$$

Since $\tilde{f}$ is a finite linear combination of indicator functions of intervals, the proof is now complete. □

Lemma 1.7. *Under assumptions of Theorem 1.6*

$$\overline{\lim_{N \to \infty}} \frac{1}{N} H(\nu^t_{N,n_N} \,|\, \mu_{N,n_N}) \leq \int_S h(m(t,\theta))\, d\theta - h(\bar{m}),$$

Proof of Lemma 1.7. Repeat the proof of Lemma 1.5 through (1.28) with k, l replaced by $k_N = \frac{1}{\varepsilon}\sqrt{N}$, $l_N = \varepsilon\sqrt{N}$ respectively for some small fixed $\varepsilon > 0$. We need to obtain an estimate from above on

$$\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} E_{\nu^s_{N,n_N}}\Big(-\frac{1}{N}\log\mu_{k_N}(\bar n_1,\dots,\bar n_{k_N})\Big)\,ds.$$

Proceeding as in Lemma 1.5 (see (1.29)) we obtain for any $s > 0$

$$\varlimsup_{N\to\infty}\bigg(-\frac{1}{N}E_{\mu_{N,n_N}} f^s_{k_N}\log\mu_{k_N}(\bar n_1,\dots,\bar n_{k_N})\bigg)$$
$$= -h(\bar m) + \varlimsup_{N\to\infty}\frac{1}{k_N}\sum_{i=1}^{k_N} E_{\nu^s_{N,n_N}}\bigg(-\frac{1}{l_N}\log\binom{l_N}{\bar n_i}\bigg). \quad (1.32)$$

We use the following consequence of Stirling's formula:

Lemma 1.8. *Let h be defined by (1.12). There exist positive constants c_1 and c_2 such that for all $n \in \mathbb{N}$ and $k \in \{0, 1, \dots, n\}$*

$$-\frac{c_1}{n}\log n \le \frac{1}{n}\log\binom{n}{k} + h\Big(\frac{k}{n}\Big) \le \frac{c_2}{n}. \quad (1.33)$$

Corollary. *For any sequence $\{k_n\}_{n=1}^{\infty}$, $k_n \in \{0, 1, \dots, n\}$,*

$$\lim_{n\to\infty}\bigg(\frac{1}{n}\log\binom{n}{k_n} + h\Big(\frac{k_n}{n}\Big)\bigg) = 0.$$

Proof of Lemma 1.8. The case when $k = 0$ or $k = n$ is trivial and we assume that $n \ge 2$ and $k \in \{1, 2, \dots, n-1\}$. By Stirling's formula (see, for example, [6], p. 73) for any $l \in \mathbb{N}$ we have

$$\frac{1}{12l+1} < \log l! - \frac{1}{2}\log(2\pi) - \Big(l+\frac{1}{2}\Big)\log l < \frac{1}{12l}.$$

Applying this inequalities with $l = n$, $(n-k)$, k after elementary computation we obtain:

$$\frac{1}{n}\bigg(\frac{1}{12n+1} - \frac{1}{12}\frac{n}{k(n-k)}\bigg) <$$
$$\frac{1}{n}\log\binom{n}{k} - \frac{1}{2n}\log(2\pi) + h\Big(\frac{k}{n}\Big) - \frac{1}{2n}\log\frac{n}{k(n-k)}$$
$$< \frac{1}{n}\bigg(\frac{1}{12n} - \frac{1}{12k+1} - \frac{1}{12(n-k)+1}\bigg),$$

which immediately implies (1.33) upon noticing that

$$\frac{4}{n} \leq \frac{n}{k(n-k)} \leq \frac{n}{n-1}.$$

□

By the corollary to Lemma 1.8 we can replace $-\frac{1}{l_N}\log\binom{l_N}{\bar{n}_i}$ with $h(\frac{\bar{n}_i}{l_N})$ in (1.32). Therefore we have shown that

$$\varlimsup_{N\to\infty} \frac{1}{N} H_N(\nu^t_{N,n_N} \mid \mu_{N,n_N}) \leq C\varepsilon + \limsup_{N\to\infty} E_{\nu^s_{N,n_N}} \frac{1}{k_N}\sum_{i=1}^{k_N} h\Big(\frac{\bar{n}_i}{l_N}\Big) - h(\bar{m}).$$

For any $x \in \Lambda_N$ define the translation operator τ_x acting on functions $f : \mathbb{X}_N \to \mathbb{R}$ by the formula $\tau_x f(\eta) = f(\tau_x \eta)$ where $(\tau_x\eta)_y = \eta_{x+y}$, $y \in \Lambda_N$. The left hand side of the above inequality does not depend on the way we divide the circle into "arcs" of length $\frac{1}{k_N}$: we need not have the first l_N sites in the first "arc". Notice also that the entropy $H_N(\nu^t_{N,n_N} \mid \mu_{N,n_N})$ is a non-increasing function of time (see Lemma 1.3). Taking this into account we arrive at the following inequality

$$\varlimsup_{N\to\infty} \frac{1}{N} H_N(\nu^t_{N,n_N} \mid \mu_{N,n_N}) \leq C\varepsilon - h(\bar{m})$$

$$+ \varliminf_{l\to\infty} \varlimsup_{N\to\infty} \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} \frac{1}{2l+1}\sum_{|x|\leq l} E_{\nu^s_{N,n_N}} \frac{1}{k_N}\sum_{i=1}^{k_N} \tau_x h\Big(\frac{\bar{n}_i}{l_N}\Big)\, ds. \quad (1.34)$$

The next lemma is the key step in the proof of Lemma 1.7.

Lemma 1.9. *Let $t_0 > 0$ and $k_N \to \infty$ in such a way that $\frac{k_N}{N} \to 0$ as $N \to \infty$ and $l_N k_N = N$. Then*

$$\varlimsup_{l\to\infty} \varlimsup_{N\to\infty} \int_{t_0}^{t} \left(\frac{1}{2l+1}\sum_{|x|\leq l} E_{\nu^s_{N,n_N}} \frac{1}{k_N}\sum_{i=1}^{k_N} \tau_x h\Big(\frac{\bar{n}_i}{l_N}\Big) - \int_S h(m(s,\theta))\, d\theta \right) ds \leq 0.$$

Proof of Lemma 1.9 This lemma is the consequence of the convexity of h and the two-block estimate which is the content of Lemma 1.10. For any configuration η define the average over a block of size $2l+1$ centered at $x \in \Lambda_N$ by

$$\bar{\eta}_{x,l} \stackrel{\text{def}}{=} \frac{1}{2l+1} \sum_{|y| \le l} \eta_{x+y}. \tag{1.35}$$

Lemma 1.10. *For any $t > 0$*

$$\lim_{\varepsilon \downarrow 0} \overline{\lim_{l \uparrow \infty}} \, \overline{\lim_{N \uparrow \infty}} \, E_N \int_0^t \frac{1}{N} \sum_{x \in \Lambda_N} |\bar{\eta}_{x,l} - \bar{\eta}_{x,\varepsilon N}| \, ds = 0$$

The proof of Lemma 1.10 is postponed till the next subsection.

Now we continue with the proof of Lemma 1.9.

Step 1. Consider the i-th "arc" of size l_N and divide it into "sub-arcs" of size $(2l+1)$. Denote the center of the j-th "sub-arc" by $x_i^{(j)}$. By convexity of h

$$h\left(\frac{\bar{n}_i}{l_N}\right) = h\left(\frac{2l+1}{l_N} \sum_{j=1}^{\frac{l_N}{2l+1}} \bar{\eta}_{x_i^{(j)},l}\right) \le \frac{2l+1}{l_N} \sum_{j=1}^{\frac{l_N}{2l+1}} h(\bar{\eta}_{x_i^{(j)},l}).$$

Fix an arbitrary $\varepsilon > 0$. Then

$$\begin{aligned} \frac{1}{2l+1} \sum_{|x| \le l} \frac{1}{k_N} \sum_{i=1}^{k_N} \tau_x h\left(\frac{\bar{n}_i}{l_N}\right) - \int\limits_S h(m(s,\theta)) \, d\theta \\ \le \frac{1}{2l+1} \sum_{|x| \le l} \frac{1}{k_N} \sum_{i=1}^{k_N} \frac{2l+1}{l_N} \sum_{j=1}^{\frac{l_N}{2l+1}} h(\tau_x \bar{\eta}_{x_i^{(j)},l}) - \int\limits_S h(m(s,\theta)) \, d\theta \\ = \frac{1}{N} \sum_{x \in \Lambda_N} \left(h(\bar{\eta}_{x,l}) - h(\bar{\eta}_{x,\varepsilon N})\right) \\ + \frac{1}{N} \sum_{x \in \Lambda_N} h(\bar{\eta}_{x,\varepsilon N}) - \int\limits_S h(m(s,\theta)) \, d\theta. \end{aligned} \tag{1.36}$$

Step 2. We apply the two-block estimate to the expectation of the first term in the right hand side of (1.36). Function h is uniformly continuous on $[0,1]$, i.e. for any $\tilde{\varepsilon} > 0$ there exists $\delta > 0$ such that

$$|h(y_1) - h(y_2)| < \tilde{\varepsilon} \quad \text{as soon as} \quad |y_1 - y_2| < \delta.$$

Since $\sup |h| \le \log 2$ we obtain

$$0 \le \int_{t_0}^{t} E_{\nu^s_{N,n_N}} \frac{1}{N} \sum_{x \in \Lambda_N} \left(h(\bar{\eta}_{x,l}) - h(\bar{\eta}_{x,\varepsilon N})\right) ds$$
$$\le \tilde{\varepsilon}(t - t_0) + \frac{2\log 2}{\delta} \int_{t_0}^{t} E_{\nu^s_{N,n_N}} \frac{1}{N} \sum_{x \in \Lambda_N} |\bar{\eta}_{x,l} - \bar{\eta}_{x,\varepsilon N}|\, ds.$$

Applying Lemma 1.10 to the last term we see that

$$\lim_{\varepsilon \downarrow 0} \overline{\lim_{l \uparrow \infty}}\, \overline{\lim_{N \uparrow \infty}} \int_{t_0}^{t} E_{\nu^s_{N,n_N}} \frac{1}{N} \sum_{x \in \Lambda_N} \left(h(\bar{\eta}_{x,l}) - h(\bar{\eta}_{x,\varepsilon N})\right) ds \le \tilde{\varepsilon}(t - t_0).$$

Since $\tilde{\varepsilon}$ was arbitrary we conclude that the limit above is equal to zero.

Step 3. To finish the proof of Lemma 1.9 it is enough to show that

$$\lim_{\varepsilon \downarrow 0} \overline{\lim_{N \uparrow \infty}}\, E_{\nu^s_{N,n_N}} \frac{1}{N} \sum_{x \in \Lambda_N} h(\bar{\eta}_{x,\varepsilon N}) = \int_S h(m(s,\theta))\, d\theta. \tag{1.37}$$

This is a consequence of the existence of the scaling limit. Indeed, let

$$m_\varepsilon(s,u) = \frac{1}{2\varepsilon} \int_{u-\varepsilon}^{u+\varepsilon} m(s,\theta)\, d\theta.$$

For any $u \in S$ and any $\delta > 0$

$$\nu^s_{N,n_N}\left(|\bar{\eta}_{[uN],\varepsilon N} - m_\varepsilon(s,u)| > \delta\right) \to 0 \quad \text{as} \quad N \to \infty.$$

Therefore

$$\lim_{N \to \infty} E_{\nu^s_{N,n_N}} \frac{1}{N} \sum_{x \in \Lambda_N} \left[h(\bar{\eta}_{x,\varepsilon N}) - h\left(m_\varepsilon\left(s, \tfrac{x}{N}\right)\right)\right] = 0,$$

and we need only to prove that

$$\lim_{\varepsilon\downarrow 0}\overline{\lim_{N\uparrow\infty}}\frac{1}{N}\sum_{x\in\Lambda_N} h\big(m_\varepsilon\big(s,\tfrac{x}{N}\big)\big) = \int\limits_S h(m(s,\theta))\,d\theta.$$

Since $m_\varepsilon(s,u)$ is a continuous function of u for any fixed s, we have that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{x\in\Lambda_N} h(m_\varepsilon(s,\tfrac{x}{N})) = \int\limits_S h(m_\varepsilon(s,u))\,du.$$

By continuity of h and Proposition 1.3 (see the next subsection)

$$\lim_{\varepsilon\downarrow 0}\int\limits_S h(m_\varepsilon(s,u))\,du = \int\limits_S h(m(s,\theta))\,d\theta.$$

The proof of Lemma 1.9 is now complete. □

We proceed with the proof of Lemma 1.7. Applying Lemma 1.9 to (1.34) we obtain

$$\overline{\lim_{N\to\infty}}\frac{1}{N}H(\nu^t_{N,n_N}\mid\mu_{N,n_N}) \le C\varepsilon - h(\bar m) + \frac{1}{\varepsilon}\int_{t-\varepsilon}^t\int\limits_S h(m(s,\theta))\,d\theta\,ds.$$

We show below (see Lemma 1.11) that the function $\int\limits_S h(m(s,\theta))\,d\theta$ is continuous in s. By letting ε go to zero we obtain the statement of Lemma 1.7. □

Lemma 1.11. *Let m be the entropic solution of (1.30) with measurable initial data m_0, $0\le m_0\le 1$. Then the function $\int_S h(m(s,\theta)\,d\theta$ is continuous in $s\in[0,\infty)$.*

Proof of Lemma 1.11 At first we notice that if $F\in C^1(S)$ then

$$\int_S F(m(s,\theta))\,d\theta$$

is continuous in s, $s \geq 0$. Let $s, t \geq 0$ then

$$\begin{aligned}\Big| \int_S F(m(t,\theta))\, d\theta - \int_S F(m(s,\theta))\, d\theta \Big| & \\ &\leq \max_{y \in S} |F'(y)| \int_S |m(t,\theta) - m(s,\theta)|\, d\theta \\ &\leq C \int_S |m(|t-s|,\theta) - m_0(\theta)|\, d\theta \to 0 \quad \text{as } |t-s| \downarrow 0.\end{aligned}$$

See, for example, Ch. 7 (relations (20.1) and (10.4)) of [4] or Ch. 16 of [15].

Function h is continuously differentiable on any set $[\delta, 1-\delta]$, $\delta > 0$. Therefore if m_0 is bounded away from 0 and 1 then the proof is complete.

For the general case we use the fact that h is continuous on $[0,1]$ and $h(0) = h(1) = 0$. This implies that the contribution of those θ for which $m(t,\theta)$ is close to 0 or 1 can be made arbitrarily small. Namely, choose an $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$ such that $|h(y)| \leq \varepsilon$ for any $y \in [0, 2\delta] \cup [1-2\delta, 1]$. Let ψ_δ be a function with the following properties:

$$\begin{aligned}&\psi_\delta \in C^\infty([0,1]), \quad 0 \leq \psi_\delta(y) \leq 1, \\ &\psi_\delta(y) = 1 \quad \text{for } \ y \in [0,\delta] \cup [1-\delta, 1], \\ &\psi_\delta(y) = 0 \quad \text{for } \ y \in [2\delta, 1-2\delta].\end{aligned}$$

We have

$$\begin{aligned}\int_S h(m(s,\theta))\, d\theta = &\int_S h(m(s,\theta))\big(1 - \psi_\delta(m(s,\theta))\big)\, d\theta \\ &+ \int_S h(m(s,\theta))\psi_\delta(m(s,\theta))\, d\theta.\end{aligned}$$

Observe that by our choice of δ

$$\int_S |h(m(s,\theta))\psi_\delta(m(s,\theta))|\, d\theta \leq \varepsilon$$

independently of s. Denote the function $h(y)(1-\varphi_\delta(y))$ by F_δ. Then $F_\delta \in C^1(S)$ and our previous reasoning applies. $\square$

1.3.3 Proof of the two-block estimate

Proof of Lemma 1.10. The proof is based on a number of lemmas. Let $j^(N)(t)$ be the difference between the total number of jumps to the right and the total number of jumps to the left up to time t. The first lemma gives us a "microscopic" description of the behavior of j_N as $N \to \infty$ for a fixed t.

Lemma 1.12. *For any* $\delta > 0$

$$\varlimsup_{N\to\infty} P_N\left(\left|\frac{j^(N)(t)}{N^2} - \frac{(p-q)}{N}\int_0^t \sum_{x\in\Lambda_N} \eta_x(s)(1-\eta_{x+1}(s))\,ds\right| > \delta\right) = 0. \quad (1.38)$$

Proof. Consider a process $(\eta^{(N)}(t), j^{(N)}(t))$ with the state space $\mathbb{X}_N \times (\mathbb{N}\cup\{0\})$ and the generator

$$\tilde{\mathcal{L}}_p f(\eta, j) = N \sum_{x\in\Lambda_N} \big[p\eta_x(1-\eta_{x+1})(f(\eta^{x,x+1}, j+1) - f(\eta, j)) \\ + q\eta_{x+1}(1-\eta_x)(f(\eta^{x,x+1}, j-1) - f(\eta, j))\big].$$

Assume that at time $t = 0$ the process starts from $(\eta^{(N)}(0), 0)$. Choose $f(\eta, j) = j$ then

$$\tilde{\mathcal{L}}_p j^{(N)} = (p-q)N \sum_{x\in\Lambda_N} \eta_x^{(N)}(1-\eta_{x+1}^{(N)}), \quad (1.39)$$

$$dj^{(N)}(t) = \tilde{\mathcal{L}}_p j^{(N)}(t)\,dt + dM_1(t), \quad (1.40)$$

and

$$dM_1^2(t) = \big[\tilde{\mathcal{L}}_p\big(j^{(N)}(t)\big)^2 - 2j^{(N)}(t)\tilde{\mathcal{L}}_p j^{(N)}(t)\big]\,dt + dM_2(t), \quad (1.41)$$

where $M_1(t)$ and $M_2(t)$ are martingales. Since

$$\tilde{\mathcal{L}}_p\big(j^{(N)}(t)\big)^2 - 2j^{(N)}(t)\tilde{\mathcal{L}}_p j^{(N)}(t) = N \sum_{x\in\Lambda_N} \eta_x^{(N)}(t)(1-\eta_{x+1}^{(N)}),$$

we obtain from (1.41) that for any $t \geq 0$

$$E_N M_1^2(t) = O(N^2). \tag{1.42}$$

Therefore by (1.40), (1.39) and (1.42) we conclude that

$$\begin{aligned} P_N\bigg(\Big|\frac{j^{(N)}(t)}{N^2} - \frac{(p-q)}{N}\int_0^t \sum_{x\in\Lambda_N} \eta_x(s)(1-\eta_{x+1}(s))\,ds\Big| > \delta\bigg) \\ = P_N\big(|M_1(t)-M_1(0)| > \delta N^2\big) \leq \frac{1}{\delta^2 N^4} E_N|M_1(t)-M_1(0)|^2 \\ \leq \frac{C(t)}{\delta^2 N^2} \to 0 \quad \text{as} \quad N\to\infty. \end{aligned}$$

□

We use the following one-block estimate to calculate the same quantity as above but now in terms of large microscopic blocks:

Lemma 1.13. *Let f be a local function on the configuration space. Then for any test function $J \in C(S)$ and any $\delta > 0$*

$$\overline{\lim_{l\to\infty}}\ \overline{\lim_{N\to\infty}}\ P_N\bigg(\frac{1}{N}\int_0^t \bigg|\sum_{x\in\Lambda_N} J\Big(\frac{x}{N}\Big)\big(\tau_x f(\eta(s)) - \bar{f}(\bar{\eta}_{x,l}(s))\big)\bigg|\,ds > \delta\bigg) = 0,$$

where (assuming that f depends on n coordinates) we set

$$\bar{f}(u) = E_{\mu_n^u} f \stackrel{\text{def}}{=} \sum_{(\alpha_1,\ldots,\alpha_n)\in\{0,1\}^n} f(\alpha_1,\ldots,\alpha_n) u^{\sum\alpha_i}(1-u)^{n-\sum\alpha_i}.$$

For the proof see, for example, [8].

Apply this lemma with $f(\eta) = \eta_0(1-\eta_1)$. Then $\bar{f}(u) = u(1-u)$ and

$$\overline{\lim_{l\to\infty}}\ \overline{\lim_{N\to\infty}}\ P_N\left(\frac{1}{N}\int_0^t \Big| \sum_{x\in\Lambda_N} \eta_x(1-\eta_{x+1}) - \sum_{x\in\Lambda_N} \bar{\eta}_{x,l}(1-\bar{\eta}_{x,l})\Big|\, ds > \delta\right) = 0. \tag{1.43}$$

From (1.38) and (1.43) we obtain

$$\overline{\lim_{l\to\infty}}\ \overline{\lim_{N\to\infty}}\ P_N\left(\left|\frac{j^{(N)}(t)}{N^2} - \frac{(p-q)}{N}\int_0^t \sum_{x\in\Lambda_N} \bar{\eta}_{x,l}(s)(1-\bar{\eta}_{x,l}(s))\, ds\right| > \delta\right) = 0. \tag{1.44}$$

The next lemma is the key step of the proof. Combined with the existence of the scaling limit it would allow us to substitute averages over small macroscopic blocks for the averages over large microscopic blocks in (1.44) and would essentially imply the two-block estimate.

Lemma 1.14. *Let m be the entropic solution of the hydrodynamic equation with the initial condition m_0. Then for any positive t and δ*

$$\overline{\lim_{N\to\infty}}\ P_N\left(\left|\frac{j^{(N)}(t)}{N^2} - (p-q)\int_0^t \int\limits_S m(s,\theta)(1-m(s,\theta))\, d\theta\, ds\right| > \delta\right) = 0.$$

The proof of this lemma will be given in the next subsection.

From the existence of the scaling limit (see Theorem 1.4) we can derive

Lemma 1.15. *For any $t > 0$*

$$\overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{N\to\infty}}\ E_N\left|\frac{1}{N}\sum_{x\in\Lambda_N} \bar{\eta}^2_{x,\varepsilon N}(t) - \int\limits_S m^2(t,\theta)\, d\theta\right| = 0.$$

Proof. Let

$$\bar{m}_\varepsilon(t,u) = \frac{1}{2\varepsilon}\int\limits_{u-\varepsilon}^{u+\varepsilon} m(t,\theta)\, d\theta.$$

We have

$$E_N\Big|\frac{1}{N}\sum_{x\in\Lambda_N}\bar\eta^2_{x,\varepsilon N}(t)-\int\limits_0^1 m^2(t,\theta)\,d\theta\Big|$$
$$\le E_N\Big|\frac{1}{N}\sum_{x\in\Lambda_N}\bar\eta^2_{x,\varepsilon N}(t)-\frac{1}{N}\sum_{x\in\Lambda_N}\bar m^2_\varepsilon(t,\frac{x}{N})\Big|$$
$$+\Big|\frac{1}{N}\sum_{x\in\Lambda_N}\bar m^2_\varepsilon(t,\frac{x}{N})-\int\limits_0^1 m^2(t,\theta)\,d\theta\Big| \quad (1.45)$$

By the existence of the hydrodynamic scaling limit

$$\lim_{N\to\infty} E_N\big|\bar\eta_{[uN],\varepsilon N}(t)-\bar m_\varepsilon(t,u)\big|=0 \qquad \text{for any } u\in S.$$

Therefore by the bounded convergence theorem the first term in the right hand side of (1.45) converges to zero as $N\to\infty$. Since $\bar m_\varepsilon(t,u)$ is continuous in u we also have that

$$\frac{1}{N}\sum_{x\in\Lambda_N}\bar m^2_\varepsilon(t,\frac{x}{N}) \;\to\; \int\limits_S \bar m^2_\varepsilon(t,\theta)\,d\theta \quad \text{as } N\to\infty.$$

Finally

$$\Big|\int\limits_0^1 \bar m^2_\varepsilon(t,\theta)\,d\theta-\int\limits_0^1 m^2(t,\theta)\,d\theta\Big|$$
$$\le 2\int\limits_0^1 |\bar m_\varepsilon(t,\theta)-m(t,\theta)|\,d\theta \;\to\; 0 \quad \text{as } \varepsilon\to 0$$

by the following

Proposition 1.3. *Let $\rho\in L^1(S)$ and*

$$\rho_\varepsilon(u)=\frac{1}{2\varepsilon}\int_{u-\varepsilon}^{u+\varepsilon}\rho(v)\,dv.$$

Then
$$\int\limits_S |\rho_\varepsilon(u) - \rho(u)|\, du \;\to\; 0 \qquad as \;\; \varepsilon \to 0.$$

This proposition is a standard fact about L^1 functions. $\square$

From (1.44), Lemma 1.14 and Lemma 1.15 it follows that

$$\overline{\lim_{\varepsilon\to 0}}\,\overline{\lim_{l\to\infty}}\,\overline{\lim_{N\to\infty}}\, P_N\left(\left|\frac{1}{N}\int_0^t \left(\sum_{x\in\Lambda_N} \bar{\eta}^2_{x,l}(s) - \sum_{x\in\Lambda_N} \bar{\eta}^2_{x,\varepsilon N}(s)\right) ds\right| > \delta\right) = 0. \tag{1.46}$$

But we need a stronger statement. The following algebraic fact about averages completes the argument:

Lemma 1.16. *For any $l \in \mathbb{N}$, sufficiently small $\varepsilon > 0$ and all N such that $\varepsilon N > l$*

$$\frac{1}{N}\sum_{x\in\Lambda_N} \left(\bar{\eta}_{x,l} - \bar{\eta}_{x,\varepsilon N}\right)^2 \le \frac{4}{N}\sum_{x\in\Lambda_N} \left(\bar{\eta}^2_{x,l} - \bar{\eta}^2_{x,\varepsilon N}\right) + \frac{C}{N} \tag{1.47}$$

where constant C depends only on l and ε.

Proof. For $l \in \mathbb{N}$ define the operator $M_l : [0,1]^N \to [0,1]^N$ by

$$(M_l\eta)_x = \bar{\eta}_{x,l}$$

where $\bar{\eta}_{x,l}$ is given by (1.35). We prove that

$$\|M_l\eta - M_{\varepsilon N}\circ M_l\eta\|^2 \le 2\left(\|M_l\eta\|^2 - \|M_{\varepsilon N}\circ M_l\eta\|^2\right). \tag{1.48}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^N$.

At first we show that the inequality (1.48) implies the statement of the lemma. We observe that

$$\left|\,(M_{\varepsilon N}\circ M_l\eta)_x - (M_{\varepsilon N}\eta)_x\,\right| \le \frac{C(\varepsilon, l)}{N}$$

uniformly in $x \in \Lambda_N$. Therefore

$$\|M_{\varepsilon N} \circ M_l \eta - M_{\varepsilon N} \eta\|^2 \le \frac{C^2(\varepsilon, l)}{N},$$

and

$$\begin{aligned}
&\frac{1}{N} \sum_{x \in \Lambda_N} (\bar{\eta}_{x,l} - \bar{\eta}_{x,\varepsilon N})^2 = \frac{1}{N} \|M_l \eta - M_{\varepsilon N} \eta\|^2 \\
&\le \frac{2}{N} \left(\|M_l \eta - M_{\varepsilon N} \circ M_l \eta\|^2 + \|M_{\varepsilon N} \circ M_l \eta - M_{\varepsilon N} \eta\|^2 \right) \\
&\le \frac{4}{N} \left(\|M_l \eta\|^2 - \|M_{\varepsilon N} \circ M_l \eta\|^2 \right) + \frac{2C^2(\varepsilon, l)}{N^2} \\
&= \frac{4}{N} \left(\|M_l \eta\|^2 - \|M_{\varepsilon N} \eta\|^2 \right) + \frac{4}{N} \left(\|M_{\varepsilon N} \eta\|^2 - \|M_{\varepsilon N} \circ M_l \eta\|^2 \right) + \frac{2C^2(\varepsilon, l)}{N^2} \\
&\le \frac{4}{N} \left(\|M_l \eta\|^2 - \|M_{\varepsilon N} \eta\|^2 \right) + \frac{8C(\varepsilon, l)}{N} + \frac{2C^2(\varepsilon, l)}{N^2}
\end{aligned}$$

as claimed.

To prove the inequality (1.48) we use the Fourier transform defined by

$$\widehat{\eta}(\lambda) = \sum_{x \in \Lambda_N} e^{ix\lambda} \eta_x, \qquad \lambda \in [0, 2\pi].$$

The function $\widehat{\eta}$ belongs $L^2(S)$ and satisfies the identity

$$\|\widehat{\eta}\|_2^2 = \int_0^{2\pi} |\widehat{\eta}(\lambda)|^2 \, d\lambda = 2\pi \|\eta\|^2.$$

Let $\chi_l : \Lambda_N \to \{0, 1\}$ be the characteristic function of the set $\{x \in \Lambda_N : |x| \le l\}$, i. e.

$$\chi_l(x) = \begin{cases} 1 & \text{if } |x| \le l, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$M_l\eta = \frac{1}{2l+1}(\chi_l * \eta);$$
$$\widehat{M_l\eta} = \frac{1}{2l+1}\widehat{\chi}_l \cdot \widehat{\eta},$$

where

$$(\chi_l * \eta)_x = \sum_{y\in\Lambda_N} \chi_l(x-y)\eta_y = \sum_{y\in\Lambda_N} \chi_l(y)\eta_{x+y}.$$

Using this notation we can write the left hand side of (1.48) as

$$\|M_l\eta - M_{\varepsilon N}\circ M_l\eta\|^2 = \frac{1}{2\pi}\|\widehat{M_l\eta} - \widehat{M_{\varepsilon N}\circ M_l\eta}\|^2$$
$$= \frac{1}{2\pi(2l+1)^2}\|\widehat{\chi}_l\widehat{\eta} - \frac{1}{2\varepsilon N+1}\widehat{\chi}_{\varepsilon N}\widehat{\chi}_l\widehat{\eta}\|^2$$
$$= \frac{1}{2\pi(2l+1)^2}\int_0^{2\pi} |\widehat{\chi}_l \cdot \widehat{\eta}(\lambda)|^2 \left(1 - \frac{1}{2\varepsilon N+1}\widehat{\chi}_{\varepsilon N}(\lambda)\right)^2 d\lambda \quad (1.49)$$

Observe that $\widehat{\chi}_{\varepsilon N}$ is real-valued. For the right hand side of (1.48) we obtain

$$\|M_l\eta\|^2 - \|M_{\varepsilon N}\circ M_l\eta\|^2$$
$$= \frac{1}{2\pi(2k+1)^2}\int_0^{2\pi} |\widehat{\chi}_l\widehat{\eta}(\lambda)|^2 \left(1 - \frac{1}{(2\varepsilon N+1)^2}\left(\widehat{\chi}_{\varepsilon N}(\lambda)\right)^2\right) d\lambda \quad (1.50)$$

Comparing (1.49) and (1.50) we see that it is enough to show that

$$f(\lambda) \stackrel{\text{def}}{=} \frac{1}{2\varepsilon N+1}\widehat{\chi}_{\varepsilon N}(\lambda)$$

is bounded away from (-1). Indeed, the inequality $f(\lambda) \geq -1+\delta$ implies that

$$1 - f(\lambda) \leq 2 - \delta \leq \frac{2-\delta}{\delta}(1 + f(\lambda))$$

and

$$(1 - f(\lambda))^2 \leq \frac{2-\delta}{\delta}(1 - f^2(\lambda)).$$

This proves (1.48) with the constant $\frac{2-\delta}{\delta}$. We show that $f(\lambda) \geq -\frac{1}{3}$ which gives us $\delta = \frac{2}{3}$ and the constant equal to 2.

It is not difficult to compute that

$$f(\lambda) = \frac{1}{2\varepsilon N + 1} \frac{\sin \lambda(2\varepsilon N + 1)/2}{\sin \lambda/2}.$$

Since $f(\lambda) = f(2\pi - \lambda),\ \lambda \in [0, \pi]$,

$$\begin{aligned}
\inf_{\lambda\in[0,2\pi]} f(\lambda) &= \inf_{\lambda\in[0,\pi]} f(\lambda) \\
&= \inf_{\lambda\in[0,\pi]} \frac{\sin \lambda(2\varepsilon N + 1)/2}{\lambda(2\varepsilon N + 1)/2} \cdot \frac{\lambda/2}{\sin \lambda/2} \\
&\geq \inf_{\lambda\in[0,\pi]} \frac{\sin \lambda(2\varepsilon N + 1)/2}{\lambda(2\varepsilon N + 1)/2} \cdot \sup_{\lambda\in[0,\pi]} \frac{\lambda/2}{\sin \lambda/2} = -\frac{2}{3\pi} \cdot \frac{\pi}{2} = -\frac{1}{3}.
\end{aligned}$$

□

Using the above lemma we can strengthen (1.46) and conclude that

$$\overline{\lim_{\varepsilon\to 0}}\, \overline{\lim_{l\to\infty}}\, \overline{\lim_{N\to\infty}}\, P_N\Big(\frac{1}{N}\int_0^t \sum_{x\in\Lambda_N} \left(\bar\eta_{x,l}(s) - \bar\eta_{x,\varepsilon N}(s)\right)^2\, ds > \delta\Big) = 0.$$

Finally by the Schwartz inequality we obtain

$$\begin{aligned}
&E_N \int_0^t \frac{1}{N} \sum_{x\in\Lambda_N} |\bar\eta_{x,l}(s) - \bar\eta_{x,\varepsilon N}(s)|\, ds \leq \\
&\qquad\qquad \sqrt{t}\Big(E_N \int_0^t \frac{1}{N} \sum_{x\in\Lambda_N} |\bar\eta_{x,l}(s) - \bar\eta_{x,\varepsilon N}(s)|^2\, ds\Big)^{\frac{1}{2}}.
\end{aligned}$$

This implies the statement of Lemma 1.10.

1.3.4 Proof of Lemma 1.14

Additivity property of $j^{(N)}(t)$ and of the Lebesgue integral implies that we may assume without loss of generality that t is small. Let $t < \frac{1}{4}$ and $\varepsilon \in (0, 1-4t)$ be fixed. Consider an arbitrary arc of length ε and color green all the particles which happened to be there at $t = 0$. At first we study the dynamics of "green" particles. We "straighten" this arc on the real line, surround it at $t = 0$ with the same environment it had on the circle and start a new process on $\mathbb{R}$. We show that for a given small t and sufficiently large N the "green" particles on $\mathbb{R}$ will not practically "feel" the difference between the line and the circle. The problem on the line allows a straightforward computation. Then we divide the circle into small arcs and apply the above procedure to each arc separately. Having done this, we rebuild the circle, putting all the arcs together, and obtain the statement of the lemma.

Step 1. (The problem on $\mathbb{R}$.) Fix an arbitrary $\tilde{\theta} \in [0, 1)$. Consider the asymmetric simple exclusion process on $\mathbb{R}$ with the following initial data: for $x \in \mathbb{Z}$ we set

$$\tilde{\eta}_x^{(N)}(0) = \begin{cases} \eta_{x(\text{mod}N)}^{(N)}(0) & \text{if } \frac{x}{N} \in [\tilde{\theta} - \frac{1}{2}, \tilde{\theta} + \frac{1}{2}), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\tilde{\eta}_x^{(N)}(0) \sim \tilde{m}_0$ where

$$\tilde{m}_0(\theta) = \begin{cases} m_0(\theta) & \text{if } \theta \in [\tilde{\theta} - \frac{1}{2}, \tilde{\theta} - \frac{1}{2}), \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\tilde{P}_N$ the probability measure corresponding to this process. For the rest of the proof all quantities related to the problem on $\mathbb{R}$ will be marked with “tilde”. By Theorem 1.4

$$\tilde\eta_x^{(N)}(t) \sim \tilde m(t,\cdot),$$

where $\tilde m$ is the entropic solution of the problem

$$\begin{aligned} &\tilde m_t + (p-q)(\tilde m(1-\tilde m))_\theta = 0, \qquad (t,\theta)\in(0,\infty)\times\mathbb{R},\\ &\tilde m\big|_{t=0} = \tilde m_0. \end{aligned} \tag{1.51}$$

Let $j^{(N)}_{\tilde\alpha_0,\tilde\beta_0}(t)$ be the difference of the total number of jumps to the right and the total number of jumps to the left made during time t by the particles which at $t=0$ belonged to the interval $[\tilde\alpha_0,\tilde\beta_0)$.

Lemma 1.17. *For any interval* $[\tilde\alpha_0,\tilde\beta_0)\subset\mathbb{R}$ *and any* $\delta>0$

$$\lim_{N\to\infty}\tilde P_N\left(\left|\frac{j^{(N)}_{\tilde\alpha_0,\tilde\beta_0}(t)}{N^2} - (p-q)\int_0^t\int_{\tilde\alpha(t)}^{\tilde\beta(t)} \tilde m(s,\theta)(1-\tilde m(s,\theta))\,d\theta\,ds\right| > \delta\right) = 0$$

where $\tilde\alpha(t)$ *and* $\tilde\beta(t)$ *are the solutions in the Filippov sense (see, for example, [13]) of the equation*

$$\frac{dv}{dt} = (p-q)\big(1-\tilde m(t,v(t))\big) \tag{1.52}$$

with initial data $\tilde\alpha_0$ *and* $\tilde\beta_0$ *respectively.*

Proof. Denote by $\tilde\alpha_N(t)$ ($\tilde\beta_N(t)$ respectively) the coordinate at time t of the rightmost particle which was below $\tilde\alpha_0$ ($\tilde\beta_0$) at $t=0$. Notice that

$$j^{(N)}_{\tilde\alpha_0,\tilde\beta_0}(t) = \sum_{\tilde\alpha_N(t)<\frac{x}{N}\le\tilde\beta_N(t)} x\,\tilde\eta_x^{(N)}(t) - \sum_{\tilde\alpha_0<\frac{x}{N}\le\tilde\beta_0} x\,\tilde\eta_x^{(N)}(0) \tag{1.53}$$

We need the following

Proposition 1.4. *Let $\tilde{\alpha}(t)$ and $\tilde{\beta}(t)$ be as in Lemma 1.17. Then for any $\delta > 0$*

$$\lim_{N\to\infty} \tilde{P}_N\left(\Big|\frac{1}{N} \sum_{\tilde{\alpha}_N(t)<\frac{x}{N}\le\tilde{\beta}_N(t)} \frac{x}{N}\,\tilde{\eta}_x(t) - \int_{\tilde{\alpha}(t)}^{\tilde{\beta}(t)} \theta\,\tilde{m}(t,\theta)\,d\theta\Big| > \delta\right) = 0.$$

Proof. It is enough to show that

$$\lim_{N\to\infty} \tilde{P}_N\left(\Big|\frac{1}{N} \sum_{\frac{x}{N}\le\tilde{\beta}_N(t)} \frac{x}{N}\,\tilde{\eta}_x(t) - \int_{-\infty}^{\tilde{\beta}(t)} \theta\,\tilde{m}(t,\theta)\,d\theta\Big| > \delta\right) = 0$$

Let

$$\chi_{(a,b)}(\theta) = \begin{cases} 1 & \text{if } b > a \text{ and } \theta \in (a,b), \\ -1 & \text{if } b < a \text{ and } \theta \in (b,a), \\ 0 & \text{else.} \end{cases}$$

The mass conservation law on the microscopic level states that

$$\frac{1}{N}\sum_{x\in\mathbb{Z}} \chi_{(-\infty,\tilde{\beta}_0)}\left(\frac{x}{N}\right)\,\tilde{\eta}_x(0) = \frac{1}{N}\sum_{x\in\mathbb{Z}} \chi_{(-\infty,\tilde{\beta}_N(t))}\left(\frac{x}{N}\right)\,\tilde{\eta}_x(t). \tag{1.54}$$

On the macroscopic level we have from (1.51) and (1.52) that

$$\int_{-\infty}^{\tilde{\beta}_0} \tilde{m}_0(\theta)\,d\theta = \int_{-\infty}^{\tilde{\beta}(t)} \tilde{m}(t,\theta)\,d\theta. \tag{1.55}$$

By our assumption on the initial data

$$\frac{1}{N}\sum_{x\in\mathbb{Z}} \chi_{(-\infty,\tilde{\beta}_N(0))}\left(\frac{x}{N}\right)\tilde{\eta}_x(0) \;\to\; \int_{-\infty}^{\tilde{\beta}_0} \tilde{m}_0(\theta)\,d\theta \quad \text{as } N\to\infty$$

and by the existence of the scaling limit

$$\tilde{P}_N\left(\Big|\frac{1}{N}\sum_{x\in\mathbb{Z}} \chi_{(-\infty,\tilde{\beta}(t))}\left(\frac{x}{N}\right)\tilde{\eta}_x(t) - \int_{-\infty}^{\tilde{\beta}(t)} \tilde{m}(t,\theta)\,d\theta\Big| > \delta\right) \;\to\; 0, \tag{1.56}$$

$$\tilde{P}_N\left(\Big|\frac{1}{N}\sum_{x\in\mathbb{Z}} \frac{x}{N}\,\chi_{(-\infty,\tilde{\beta}(t))}\left(\frac{x}{N}\right)\tilde{\eta}_x(t) - \int_{-\infty}^{\tilde{\beta}(t)} \theta\,\tilde{m}(t,\theta)\,d\theta\Big| > \delta\right) \;\to\; 0 \tag{1.57}$$

as $N \to \infty$. The relations (1.56), (1.54) and (1.55) imply that

$$\tilde{P}_N\left(\frac{1}{N}\sum_{x\in\mathbb{Z}}\left|\chi_{(\tilde{\beta}_N(t),\tilde{\beta}(t))}\left(\frac{x}{N}\right)\right|\tilde{\eta}_x(t) > \delta\right) \to 0 \quad \text{as } N \to \infty. \tag{1.58}$$

We are interested in the behavior of

$$\frac{1}{N}\sum_{x\in\mathbb{Z}}\frac{x}{N}\,\chi_{(-\infty,\tilde{\beta}_N(t))}\left(\frac{x}{N}\right)\tilde{\eta}_x(t).$$

It follows from (1.57) that it is enough to prove that

$$\tilde{E}_N\left|\frac{1}{N}\sum_{x\in\mathbb{Z}}\frac{x}{N}\,\chi_{(\tilde{\beta}_N(t),\tilde{\beta}(t))}\left(\frac{x}{N}\right)\tilde{\eta}_x(t)\right| \to 0 \qquad \text{as } N \to \infty.$$

We have

$$\begin{aligned}\left|\frac{1}{N}\sum_{x\in\mathbb{Z}}\frac{x}{N}\,\chi_{(\tilde{\beta}_N(t),\tilde{\beta}(t))}\tilde{\eta}_x(t)\right| &\le \frac{1}{N}\sum_{x\in\mathbb{Z}}\left|\frac{x}{N}\,\chi_{(\tilde{\beta}_N(t),\tilde{\beta}(t))}\right|\tilde{\eta}_x(t)\\ &\le \max\{|\tilde{\beta}_N(t)|,|\tilde{\beta}(t)|\}\frac{1}{N}\sum_{x\in\mathbb{Z}}|\chi_{(\tilde{\beta}_N(t),\tilde{\beta}(t))}|\tilde{\eta}_x(t)\end{aligned}$$

and

$$\begin{aligned}\tilde{E}_N\left|\frac{1}{N}\sum_{x\in\mathbb{Z}}\frac{x}{N}\chi_{(\tilde{\beta}_N(t),\tilde{\beta}(t))}\left(\frac{x}{N}\right)\tilde{\eta}_x(t)\right| &\le \left(\tilde{E}_N\big(\max\{|\tilde{\beta}_N(t)|,|\tilde{\beta}(t)|\}\big)^2\right)^{\frac{1}{2}}\\ &\times\left(\tilde{E}_N\Big(\frac{1}{N}\sum_{x\in\mathbb{Z}}|\chi(\tilde{\beta}_N(t),\tilde{\beta}(t))\left(\frac{x}{N}\right)|\,\tilde{\eta}_x(t)\Big)^2\right)^{\frac{1}{2}}.\end{aligned}$$

The second term in the right hand side converges to zero by (1.58) and the bounded convergence theorem. To estimate the first term we consider the movement of the particle whose coordinate we denoted by $\tilde{\beta}_N(\cdot)$. The distance it covered up to time t, i.e. $|\tilde{\beta}_N(t) - \tilde{\beta}_0|$, is bounded above by the distance it could have covered in the absence of other particles if it were to jump always to the right. In the latter situation the number of jumps has a

Poisson distribution with parameter tN. Since $|\tilde\beta_N(t) - \tilde\beta_0|$ is equal to the number of jumps divided by N, we conclude that

$$\tilde E_N |\tilde\beta_N(t) - \tilde\beta_0|^2 \le C$$

where C is a constant independent of N. □

From (1.53) and the above proposition we deduce that

$$\lim_{N\to\infty} \tilde P_N \left(\left| \frac{j^{(N)}_{\tilde\alpha_0,\tilde\beta_0}(t)}{N^2} - \left(\int_{\tilde\alpha(t)}^{\tilde\beta(t)} \theta\, \tilde m(t,\theta)\, d\theta - \int_{\tilde\alpha_0}^{\tilde\beta_0} \theta\, \tilde m_0(\theta)\, d\theta \right) \right| > \delta \right) = 0.$$

The next statement is a simple fact about solutions of (1.51). It completes the proof of Lemma 1.17.

Proposition 1.5. *Let $\tilde m$ be a weak solution of the problem (1.51). Then for any $\tilde\alpha \in \mathbb{R}$*

$$\int_0^{\tilde\alpha} \theta\, \tilde m(t,\theta)\, d\theta - \int_0^{\tilde\alpha} \theta\, \tilde m_0(\theta)\, d\theta =$$
$$(p-q)\Big(\int_0^t \int_0^{\tilde\alpha} \tilde m(s,\theta)(1-\tilde m(s,\theta))\, d\theta\, ds - \tilde\alpha \int_0^t \tilde m(s,\tilde\alpha)(1-\tilde m(s,\tilde\alpha))\, ds \Big).$$

Moreover if $\tilde\alpha(t)$ is a solution of (1.52) in the Filippov sense with the initial condition $\tilde\alpha_0$ then

$$\int_0^{\tilde\alpha(t)} \theta\, \tilde m(t,\theta)\, d\theta - \int_0^{\tilde\alpha_0} \theta\, \tilde m_0(\theta)\, d\theta =$$
$$(p-q) \int_0^t \int_0^{\tilde\alpha(s)} \tilde m(s,\theta)(1-\tilde m(s,\theta))\, d\theta\, ds.$$

Proof. Since $\tilde m$ is a solution of (1.51) then for any $\varphi,\ \psi \in C_0^\infty$ we have the equation

$$\iint \tilde m(s,\theta)\varphi(\theta)\psi'(s)\, d\theta\, ds =$$
$$-(p-q) \iint \tilde m(s,\theta)(1-\tilde m(s,\theta))\varphi'(\theta)\psi(s)\, d\theta\, ds.$$

which implies that (in the sense of distributions)

$$(p-q)\,\frac{d}{d\theta}\int \tilde{m}(s,\theta)(1-\tilde{m}(s,\theta))\psi(s)\,ds = \int \tilde{m}(s,\theta)\psi'(\theta)\,ds \in L^\infty \quad (1.59)$$

and $\int \tilde{m}(s,\theta)(1-\tilde{m}(s,\theta))\psi(s)\,ds$ is Lipschitz continuous in θ. Similarly we conclude that $\int \tilde{m}(s,\theta)\varphi(\theta)\,d\theta$ is Lipschitz continuous in s.

Let $\varphi_n \in C_0^\infty$ be a sequence such that

$$\varphi_n(\theta) \to \theta \cdot 1[0,\tilde{\alpha}](\theta) \qquad \text{in } L^1,$$
$$\varphi_n' \to 1[0,\tilde{\alpha}] - \tilde{\alpha}\,\delta(\tilde{\alpha}) \qquad \text{in } \mathcal{D}'$$

Since the function $\int \tilde{m}(s,\theta)(1-\tilde{m}(s,\theta))\psi(s)\,ds$ is continuous then

$$\lim_{n\to\infty}\int \varphi_n'(\theta)\int \tilde{m}(s,\theta)(1-\tilde{m}(s,\theta))\psi(s)\,ds\,d\theta =$$
$$\int_0^{\tilde{\alpha}}\int \tilde{m}(s,\theta)(1-\tilde{m}(s,\theta))\psi(s)\,ds\,d\theta - \tilde{\alpha}\int \tilde{m}(s,\alpha)(1-\tilde{m}(s,\alpha))\psi(s)\,ds$$

On the other hand by (1.59)

$$\lim_{n\to\infty}\int (p-q)\,\varphi_n'(\theta)\int \tilde{m}(s,\theta)(1-\tilde{m}(s,\theta))\psi(s)\,ds\,d\theta =$$
$$-\lim_{n\to\infty}\int \varphi_n(\theta)\int \tilde{m}(s,\theta)\psi'(s)\,ds\,d\theta = -\int_0^{\tilde{\alpha}}\int \theta\,\tilde{m}(s,\theta)\psi'(s)\,ds\,d\theta.$$

Therefore

$$-\int_0^{\tilde{\alpha}}\int \theta\,\tilde{m}(s,\theta)\psi'(s)\,ds\,d\theta =$$
$$(p-q)\int_0^{\tilde{\alpha}}\int \tilde{m}(s,\theta)(1-\tilde{m}(s,\theta))\psi(s)\,ds\,d\theta -$$
$$(p-q)\,\tilde{\alpha}\int \tilde{m}(s,\alpha)(1-\tilde{m}(s,\alpha))\psi(s)\,ds. \quad (1.60)$$

Choosing now a sequence $\psi_n \in C_0^\infty$ in such a way that

$$\psi_n(s) \to 1_{[0,t]}(s) \qquad \text{in } L^1,$$
$$\psi_n' \to \delta(0) - \delta(t) \qquad \text{in } \mathcal{D}'$$

we obtain from (1.60)

$$\int_0^{\tilde\alpha} \theta\, \tilde m(t,\theta)\, d\theta - \int_0^{\tilde\alpha} \theta \tilde m_0(\theta)\, d\theta = \\ (p-q)\int_0^t \int_0^{\tilde\alpha} \tilde m(s,\theta)(1-\tilde m(s,\theta))\, d\theta\, ds - \\ (p-q)\tilde\alpha \int_0^t \tilde m(s,\alpha)(1-\tilde m(s,\alpha))\, ds \quad (1.61)$$

which is the first part of the proposition.

Let $F(s,\tilde\alpha) = \int_0^{\tilde\alpha} \theta\, \tilde m(s,\theta)\, d\theta$. Then by (1.61) it is absolutely continuous in $\tilde\alpha$ and s and

$$\frac{\partial F(\tilde\alpha, s)}{\partial s} = (p-q)\Big(\int_0^{\tilde\alpha} \tilde m(s,\theta)(1-\tilde m(s,\theta))\, d\theta - \tilde\alpha \tilde m(s,\tilde\alpha)(1-\tilde m(s,\tilde\alpha))\Big).$$

Consider $F(s,\tilde\alpha(s))$, where $\tilde\alpha(s)$ is a solution of (1.52) in the Filippov sense with the initial condition $\tilde\alpha_0$. Then

$$\frac{dF(s,\tilde\alpha(s))}{ds} = \frac{\partial F(s,\tilde\alpha(s))}{\partial\tilde\alpha}\frac{d\tilde\alpha(s)}{ds} + \frac{\partial F(s,\tilde\alpha(s))}{\partial s} \\ = (p-q)\int_0^{\tilde\alpha(s)} \tilde m(s,\theta)(1-\tilde m(s,\theta))\, d\theta.$$

Integration of the above identity from 0 to t gives the second statement of the proposition. □

Step 2. Choose $\tilde\alpha_0 = \tilde\theta - \frac{\varepsilon}{2}$ and $\tilde\beta_0 = \tilde\theta + \frac{\varepsilon}{2}$. Let θ, α_0 and β_0 be the corresponding points on the circle:

$$\theta = \tilde\theta, \qquad \alpha_0 = \tilde\alpha_0 \pmod 1, \qquad \beta_0 = \tilde\beta_0 \pmod 1.$$

Since the solutions of (1.51) have bounded domain of dependence on the initial data we have that for all sufficiently small t ($t < \frac{1}{4}$)

$$\int_0^t \int_{\tilde{\alpha}}^{\tilde{\beta}} \tilde{m}(s,\theta)(1-\tilde{m}(s,\theta))\, d\theta\, ds = \int_0^t \int_{\alpha}^{\beta} m(s,\theta)(1-m(s,\theta))\, d\theta\, ds$$

where $\alpha(s)$ and $\beta(s)$ are the solutions of the equation

$$\frac{dv}{dt} = (p-q)\big(1 - m(t, v(t))\big) \tag{1.62}$$

on the circle with initial conditions α_0 and β_0 respectively.

We couple the process on the circle with the process on the line described in Step 1. The rightmost (respectively, leftmost) particle on the line can always accomplish a jump to the right (left). This might not be so for the corresponding particles on the circle. If a particle on the circle can not jump while the particle on the line attached to it can, we decouple them. As t increases, this decoupling spreads from the boundary of $[\tilde{\theta} - \frac{1}{2}, \tilde{\theta} + \frac{1}{2}) \subset \mathbb{R}$ to interior particles. Let τ_N be the first time when a particle which initially belonged to the arc $[\alpha_0, \beta_0) \in S$ is decoupled.

Lemma 1.18. *$P_N(\tau_N \leq t) \to 0$ as $N \to \infty$ for any $t < \frac{1}{4}$.*

Proof. If the arc $[\alpha_0, \beta_0)$ does not contain initially any particles then there is nothing to prove. Assume that at time 0 we colored green all the particles which belonged to the arc $[\alpha_0, \beta_0)$ and marked with letter "r" (respectively,"l") the particle on the circle attached to the rightmost (respectively, leftmost) particle on the line. All particles initially are considered to be healthy. But "r" (respectively, "l") particle becomes infected and turns black when it attempts to jump to a site occupied by the "l" ("r") particle. Since then any healthy particle becomes infected and turns black if it tries to jump to a site occupied by a black particle. Nothing happens if a black particle attempts to jump to a site occupied by a healthy particle. Then τ_N is equal to the first time when a green particle becomes infected. Clearly the first infected green particle has to be either the rightmost ("rg") or the leftmost ("lg") green particle. To obtain an estimate on τ_N we simplify the problem by keeping only two pairs of particles on the circle: "r" and "lg", and "l" and "rg". Moreover we assume that all four particles are independent, the leftmost particles can only jump to the left, the rightmost particles can only jump to the right, and the number of jumps made by each particle during time t has the Poisson distribution with parameter Nt. It is clear that

$$\tau_N \geq \min\{\text{the time when "r" meets "lg", the time when "l" meets "rg"}\}$$

and therefore

$$P_N(\tau_N > t) \geq \left(\text{Prob}\left(X < \tfrac{N(1-\varepsilon)}{2}\right)\right)^2$$

where X is a Poisson random variable with parameter $2tN$. Since X can be thought of as the sum of $2N$ independent Poisson random variables with parameter t, by the law of large numbers

$$\text{Prob}\Big(X < \tfrac{N(1-\varepsilon)}{2}\Big) = \text{Prob}\Big(\tfrac{X}{2N} < \tfrac{(1-\varepsilon)}{4}\Big) \to 1$$

as $N \to \infty$ if $\frac{(1-\varepsilon)}{4} > t$. This explains our choice of t and ε in the beginning of the proof of Lemma 1.14. $\square$

Let A_N be the event that

$$\left| \frac{j^{(N)}_{\alpha_0,\beta_0}(t)}{N^2} - (p-q)\int_0^t \int_{\alpha(s)}^{\beta(s)} m(s,\theta)(1-m(s,\theta))\, d\theta\, ds \right| > \delta.$$

Then

$$\begin{aligned} P_N(A_N) &= P_N(A_N \cap \{\tau_N \le t\}) + P_N(A_N \cap \{\tau_N > t\}) \\ &\le P_N(\tau_N \le t) + \tilde{P}_N(\tilde{A}_N) \to 0 \quad \text{as } N \to \infty, \end{aligned} \tag{1.63}$$

where $\tilde{A}_N$ is the corresponding event for the process on the line.

Step 3. Divide the circle into equal parts of length $\varepsilon < (1-4t)$. We obtain $k = \frac{1}{\varepsilon}$ arcs $[\alpha_1, \alpha_2),\ [\alpha_2, \alpha_3),\ \ldots,\ [\alpha_k, \alpha_{k+1})$, where $\alpha_i = (i-1)\varepsilon$, $i = 1, 2, \ldots, k$. Applying (1.63) with A_N constructed for $\alpha_0 = \alpha_i$ and $\beta_0 = \alpha_{i+1},\ i = 1, 2 \ldots, k$, we conclude that

$$P_N\left(\left|\frac{j^{(N)}(t)}{N^2} - (p-q)\int_0^t\int_0^1 m(s,\theta)(1-m(s,\theta))\,d\theta\,ds\right| > \delta\right) \le$$

$$\sum_{i=1}^k P_N\left(\left|\frac{j^{(N)}_{\alpha_i,\alpha_{i+1}}(t)}{N^2} - (p-q)\int_0^t\int_{\alpha_i(s)}^{\alpha_{i+1}(s)} m(s,\theta)(1-m(s,\theta))\,d\theta\,ds\right| > \frac{\delta}{k}\right) \to 0$$

as $N \to \infty$. Here $\alpha_i(s)$ is the solution of (1.62) with the initial condition α_i, $i = 1, 2, \ldots, k$. □

1.3.5 Lower bound for the specific entropy

Lemma 1.19. *Under assumptions of Theorem 1.6*

$$\varliminf_{N\to\infty} \frac{1}{N} H(\nu^t_{N,n_N} \,|\, \mu_{N,n_N}) \ge \int_0^1 h(m(t,\theta))\,d\theta - h(\bar{m}).$$

Proof. Let $\mu_N^{\frac{1}{2}}$ be a product measure defined by (1.5). Then

$$\frac{1}{N} H(\nu^t_{N,n_N} \,|\, \mu_{N,n_N}) = \frac{1}{N} H(\nu^t_{N,n_N} \,|\, \mu_N^{\frac{1}{2}}) + \frac{1}{N} E_{\nu^t_{N,n_N}} \log \frac{\mu_N^{\frac{1}{2}}(\eta)}{\mu_{N,n_N}(\eta)}.$$

Since $\frac{n_N}{N}$ converges to $\bar{m}$ as $N \to \infty$, by (1.16) we have that

$$\lim_{N\to\infty} \frac{1}{N} E_{\nu^t_{N,n_N}} \log \frac{\mu_N^{\frac{1}{2}}(\eta)}{\mu_{N,n_N}(\eta)} = \lim_{N\to\infty} \frac{1}{N} \log \frac{\binom{N}{n_N}}{2^N} = -\log 2 - h(\bar{m}).$$

The following general lemma completes the proof.

Lemma 1.20. *Let α_N be a sequence of probability measures on $\mathbb{X}_N$ such that for any $\delta > 0$ and any $J \in C(S)$*

$$\lim_{N\to\infty} \alpha_N \Big(\eta \in \mathbb{X}_N \, : \, \Big| \frac{1}{N} \sum_{x\in\Lambda_N} J\Big(\frac{x}{N}\Big)\eta_x - \int_0^1 J(\theta) m(\theta)\, d\theta \Big| > \delta \Big) = 0 \quad (1.64)$$

for some measurable function m, $0 \le m \le 1$. Then

$$\varliminf_{N\to\infty} \frac{1}{N} H(\alpha_N \,|\, \mu_N^{\frac{1}{2}}) \ge \int_0^1 h(m(\theta))\, d\theta + \log 2.$$

Proof. By the entropy inequality for any $J \in C(S)$ we have that

$$\frac{1}{N} H(\alpha_N \,|\, \mu_N^{\frac{1}{2}}) \ge \frac{1}{N} E_{\alpha_N} J\Big(\frac{x}{N}\Big)\eta_x - \frac{1}{N} \log E_{\mu_N^{\frac{1}{2}}} \exp\Big(\sum_{x\in\Lambda_N} J\Big(\frac{x}{N}\Big)\eta_x \Big).$$

Using the convergence (1.64) we obtain

$$\varliminf_{N\to\infty} \frac{1}{N} H(\alpha_N \,|\, \mu_N^{\frac{1}{2}}) \ge \sup_{J\in C(S)} \Big(\int_0^1 J(\theta) m(\theta) - \log\Big(1 + e^{J(\theta)}\Big)\, d\theta \Big) + \log 2.$$

We have to show that

$$\sup_{J\in C(S)} \mathcal{F}(J) \stackrel{\text{def}}{=} \sup_{J\in C(S)} \Big(\int_0^1 J(\theta) m(\theta) - \log\Big(1 + e^{J(\theta)}\Big)\, d\theta \Big) = \int_0^1 h(m(\theta))\, d\theta.$$

Clearly,

$$\mathcal{F}(J) \le \int_0^1 \sup_{y\in\mathbb{R}} \Big(y m(\theta) - \log\big(1 + e^y\big) \Big)\, d\theta = \int_0^1 h(m(\theta))\, d\theta.$$

where the supremum is attained at

$$y(\theta) = \begin{cases} \log \dfrac{m(\theta)}{1-m(\theta)} & \text{if } m(\theta) \neq 0, 1; \\ -\infty & \text{if } m(\theta) = 0; \\ +\infty & \text{if } m(\theta) = 1. \end{cases}$$

Therefore, if m were continuous and did not take values 0 and 1, the function $J_*(\theta) = y(\theta)$ would give us a solution of the variational problem in question. For a general m we need an additional argument to finish the proof.

Let m_n be a sequence of continuous functions between 0 and 1 such that $m_n \to m$ as $n \to \infty$ a.e. on S. By Egorov's theorem for any $\varepsilon > 0$ there is a set E_ε such that

(i) $\operatorname{meas} E_\varepsilon \geq 1 - \varepsilon$;

(ii) $m_n \to m$ uniformly on E_ε as $n \to \infty$.

For any $\varepsilon > 0$ define

$$m_{n,\varepsilon}(\theta) = \begin{cases} m_n(\theta) & \text{if } \ \varepsilon < m_n(\theta) < 1-\varepsilon; \\ \varepsilon & \text{if } \ m_n(\theta) \leq \varepsilon; \\ 1-\varepsilon & \text{if } \ m_n(\theta) \geq 1-\varepsilon. \end{cases}$$

Functions $m_{n,\varepsilon}$ are continuous on S. Let

$$J_{n,\varepsilon}(\theta) = \log \frac{m_{n,\varepsilon}(\theta)}{1 - m_{n,\varepsilon}(\theta)}.$$

Functions $J_{n,\varepsilon}$ are continuous on S and $|J_{n,\varepsilon}| < 2|\log \varepsilon|$ uniformly in n. We claim that

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathcal{F}(J_{n,\varepsilon}) = \int_0^1 h(m(\theta))\, d\theta. \tag{1.65}$$

The proof is straightforward. For an arbitrary $\varepsilon > 0$, all $n \geq n_0(\varepsilon)$ and all $\theta \in E_\varepsilon$ we have that

$$|m_{n,\varepsilon}(\theta) - m(\theta)| \leq |m_{n,\varepsilon}(\theta) - m_n(\theta)| + |m_n(\theta) - m(\theta)| < 2\varepsilon$$

We write

$$\begin{aligned} |\mathcal{F}(J_{n,\varepsilon}) - \int_0^1 h(m(\theta))\, d\theta| &\leq |\mathcal{F}(J_{n,\varepsilon}) - \int_0^1 h(m_{n,\varepsilon}(\theta))\, d\theta| \\ &\quad + |\int_0^1 h(m_{n,\varepsilon}(\theta))\, d\theta - \int_0^1 h(m(\theta))\, d\theta| = I_1 + I_2. \end{aligned}$$

Estimating I_1 and I_2 we find that

$$\begin{aligned} I_1 = \left| \int_0^1 J_{n,\varepsilon}(\theta)(m_{n,\varepsilon}(\theta) - m(\theta))\, d\theta \right| &\leq \left(\int_{E_\varepsilon} + \int_{S \setminus E_\varepsilon} \right) |J_{n,\varepsilon}||m_{n,\varepsilon} - m|\, d\theta \\ &\leq 2|\log \varepsilon|(2\varepsilon + \varepsilon) = 6\varepsilon|\log \varepsilon| \end{aligned}$$

and

$$I_2 \leq \left(\int_{E_\varepsilon} + \int_{S \setminus E_\varepsilon} \right) |h(m_{n,\varepsilon}(\theta)) - h(m(\theta))|\, d\theta \leq |h(2\varepsilon)| + \varepsilon \log 2.$$

Finally by letting ε go to zero we obtain (1.65).

Chapter 2

Ginzburg-Landau model

2.1 Description of the model

Let Λ_N be the one-dimensional periodic lattice $\mathbb{Z}/N\mathbb{Z}$. The variable $x_i \in \mathbb{R}$, $i = 1, \ldots, N$, attached to a lattice site, represents a "charge" at this site. The collection of all "charges" is a vector $x^{(N)}$ in $\mathbb{R}^N$ which we call the configuration. The nearest neighbor charges interact and the configuration $x^{(N)}$ changes in time. Applying the diffusive scaling of space and time, i.e. shrinking the spacing between "charges" by N and speeding up the time by N^2, we obtain a system of "charges" located at points $\frac{i}{N}$, $i = 1, \ldots, N$, of the circle $S = \mathbb{R}/\mathbb{Z}$. The evolution of $x^{(N)}(t)$ is described by the following system of stochastic differential equations:

$$dx_i(t) = \frac{N^2}{2} \left[\varphi'(x_{i+1}(t)) - 2\varphi'(x_i(t)) + \varphi'(x_{i-1}(t))\right] dt \qquad (2.1)$$
$$+ N \left(d\beta_i(t) - d\beta_{i+1}(t)\right), \quad i \in \Lambda_N,$$

where $\beta_i,\ i = 1, \ldots, N$ are independent Brownian motions and $\varphi \in C^2(\mathbb{R})$ satisfies the conditions $(H_1) - (H_3)$ listed below.

(H_1) $\int\limits_{\mathbb{R}} e^{-\varphi(x)} dx = 1.$

(H_2) There exist constants $C_0 > 0,\ C_1 \geq 0$ and $p \in [1/2, 1]$ such that $C_0 \leq \varphi''(x) \leq C_1(\varphi(x) + 1)$ and $|\varphi'(x)| \leq C_1(|\varphi(x)|^p + 1)$ for all $x \in \mathbb{R}$.

(H_3) $\int\limits_{\mathbb{R}} e^{\alpha|\varphi'(x)| - \varphi(x)} dx < \infty$ for any $\alpha > 0$.

For a simple example one can take $\varphi(x) = \frac{x^2}{2} - \frac{1}{2}\log(2\pi)$.

Notice that the convexity assumption in (H_2) guarantees the convergence of the following integral:

(H_2') $\int\limits_{\mathbb{R}} e^{\lambda x - \varphi(x)} dx \stackrel{\text{def}}{=} M(\lambda) < \infty$ for any $\lambda \in \mathbb{R}$.

We also assume that at $t = 0$ the system starts from a deterministic configuration: $x^{(N)}(0) = \eta^{(N)} = (\eta_1^{(N)}, \ldots, \eta_N^{(N)})$, and the sequence $\eta^{(N)},\ N = 1, 2, \ldots$ possesses a macroscopic profile $m_0 \in L^1(S)$. That is for any smooth function J on S there is a limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} J\left(\frac{i}{N}\right) \eta_i^{(N)} = \int_0^1 J(\theta) m_0(\theta) d\theta. \tag{2.2}$$

In particular,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \eta_i^{(N)} = \int_0^1 m_0(\theta) d\theta \stackrel{\text{def}}{=} \bar{m}. \tag{2.3}$$

Notice that the total charge $\sum\limits_{i=1}^{N} x_i$ is preserved under the dynamics defined by the system (2.1).

Sometimes we find it convenient to write the system (2.1) using a matrix notation:

$$dx(t) = N^2 A^{(N)} b^{(N)}(x(t)) dt + N G^{(N)} d\beta^{(N)}. \tag{2.4}$$

Here $\beta^{(N)}$ is a standard N-dimensional Brownian motion, $A^{(N)} = G^{(N)}G^{(N)*}$ and

$$G^{(N)} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \\ -1 & 0 & 0 & \dots & 1 \end{pmatrix}, \qquad b^{(N)}(x) = -\frac{1}{2}\begin{pmatrix} \varphi'(x_1) \\ \varphi'(x_2) \\ \dots \\ \dots \\ \varphi'(x_N). \end{pmatrix} \tag{2.5}$$

The corresponding measure on the space of continuous paths will be denoted by P_N.

The infinitesimal generator of the diffusion process $x^{(N)}(t)$ is given by

$$\begin{aligned} \mathcal{L}_N = \frac{N^2}{2} \sum_{i\in\Lambda_N} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 \qquad (2.6) \\ - \frac{N^2}{2} \sum_{i\in\Lambda_N} (\varphi'(x_i) - \varphi'(x_{i+1})) \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right). \end{aligned}$$

It is formally symmetric with respect to the probability measure

$$d\mu_N(x) = \exp(-\sum_{i=1}^{N} \varphi(x_i))dx_1 \dots dx_N \tag{2.7}$$

on $\mathbb{R}^N$ which is an invariant but non-ergodic measure for our diffusion. Indeed, any set of the form $\{x \in \mathbb{R}^N : \ y_1 < \frac{1}{N}\sum_{i=1}^{N} x_i < y_2\},\ y_1 < y_2$, is invariant but its μ_N-measure is strictly between 0 and 1. This reflects the fact that $\mathcal{L}_N$ is degenerate in the direction of vector $(1, 1, \dots, 1)$. Let $\mathcal{L}_{N,y}$ be the restriction of $\mathcal{L}_N$ to the hyperplane

$$\pi_N(y) = \{x \in \mathbb{R}^N : \ \sum_{i=1}^{N} x_i = Ny\}. \tag{2.8}$$

The operator $\mathcal{L}_{N,y}$ is strictly elliptic for any $y \in \mathbb{R}$. If we consider now a conditional measure $\mu_{N,y}(\cdot) = \mu_N(\cdot \,|\, \sum_{i=1}^{N} x_i^{(N)} = Ny)$ supported on $\pi_N(y)$, then for any $y \in \mathbb{R}$ the measure $\mu_{N,y}$ is invariant and ergodic. It is absolutely continuous with respect to the Lebesgue measure $d\sigma_N$ on the hyperplane $\pi_N(y)$ and its density is given by

$$\frac{d\mu_{N,y}}{d\sigma_N} = \frac{\exp(-\sum_{i=1}^{N} \varphi(x_i))}{\int_{\pi_N(y)} \exp(-\sum_{i=1}^{N} \varphi(x_i)) d\sigma_N}, \quad x \in \pi_N(y).$$

The Dirichlét form

$$D_{N,y}(h) \stackrel{\text{def}}{=} -\int_{\pi_N(y)} h\mathcal{L}_{N,y}h \, d\mu_{N,y} \tag{2.9}$$

$$= \frac{N^2}{2} \int_{\pi_N(y)} \sum_{i=1}^{N} \left[\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right) h \right]^2 d\mu_{N,y} \tag{2.10}$$

is non-degenerate for all $h \in W^{1,2}(\pi_N(y), d\mu_{N,y})$, $h \neq const$ a.s..

2.2 Entropy and the existence of the hydrodynamic limit

Let ν_N^t be the distribution of charges at time $t > 0$ in $\mathbb{R}^N$ corresponding to the deterministic initial distribution $\eta^{(N)}$. Due to the conservation law it is supported on the hyperplane $\pi_N(y_N)$ where $y_N = \frac{1}{N} \sum_{i=1}^{N} \eta_i^{(N)}$. Restricting ν_N^t on $\pi_N(y_N)$ we obtain a measure ν_{N,y_N}^t which is absolutely continuous with respect to μ_{N,y_N}. Consider the following Cauchy problem

$$\begin{aligned} \frac{\partial f}{\partial t} &= \mathcal{L}_N f \quad \text{on} \quad (0,+\infty)\times \mathbb{R}^N \\ f\Big|_{t=0} &= \delta_{\eta^{(N)}}. \end{aligned}$$

This problem has a unique solution f_N^t which is supported on $\pi_N(y_N)$ and

$$f^t_{N,y_N} = \frac{d\nu^t_{N,y_N}}{d\mu_{N,y_N}} = \frac{f^t_N\big|_{\pi_N(y_N)}}{\int\limits_{\pi_N(y_N)} \exp(-\sum\limits_{i=1}^N \varphi(x_i))d\sigma_N}.$$

It is a consequence of ellipticity of $\mathcal{L}_{N,y_N}$ that f^t_{N,y_N} is smooth and strictly positive function on $\pi_N(y_N)$ for all $t>0$.

Define the specific microscopic entropy $H_{N,y_N}(t)$ by the formula

$$\begin{aligned} H_{N,y_N}(t) &= \frac{1}{N} H(\nu^t_{N,y_N}|\mu_{N,y_N}) = \frac{1}{N}\int\limits_{\pi_N(y_N)} \log\left(\frac{d\nu^t_{N,y_N}}{d\mu_{N,y_N}}\right) d\nu^t_{N,y_N} \\ &= \frac{1}{N}\int\limits_{\pi_N(y_N)} f^t_{N,y_N}\log f^t_{N,y_N} d\mu_{N,y_N}. \end{aligned} \tag{2.11}$$

The function $H_{N,y_N}(t)$ is decreasing:

$$\begin{aligned} \frac{dH_{N,y_N}(t)}{dt} &= -\frac{N}{2}\int\limits_{\pi_N(y_N)} \frac{1}{f^t_{N,y_N}} \sum_{i\in\Lambda_N}\left[\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_{i+1}}\right) f^t_{N,y_N}\right]^2 d\mu_{N,y_N} \\ &= -\frac{4}{N} D_{N,y_N}\left(\sqrt{f^t_{N,y_N}}\right) < 0. \end{aligned} \tag{2.12}$$

It is known (see [11]) that if φ satisfies $(H_1)-(H_3)$ and there is a constant $C>0$ such that $\frac{1}{N}\sum\limits_{i=1}^N \varphi(\eta_i^{(N)}) \le C$, then

$$H_{N,y_N}(t) \le C(t) < \infty \tag{2.13}$$

for any $t>0$. Moreover, if we choose a sequence $\{t_N\}$, $\frac{const}{N^2} \le t_N \to 0$ then the corresponding constants $C(t_N)$ in (2.13) are still uniformly bounded in N.

This entropy estimate allows one to extend the method of [7] to the case of deterministic initial data and prove that for all $t > 0$ the system possesses a macroscopic profile $m(t, \theta)$, $\theta \in S$ to which it converges in the weak sense as $N \to \infty$. We shall formulate this statement more precisely after introducing some notation.

Let M be defined by (H_2'). It is easy to check that

$$\rho = \log M(\lambda) \tag{2.14}$$

is a convex analytic function. Its convex conjugate

$$h(y) = \sup_{z\in\mathbb{R}}(yz - \log M(z)) \tag{2.15}$$

is also analytic and

$$h'^{-1}(\lambda) = \frac{M'(\lambda)}{M(\lambda)} = \rho'(\lambda). \tag{2.16}$$

The following theorem was proven in [11].

Theorem 2.1. *Let the sequence $\eta^{(N)}$ of deterministic configurations satisfy the condition (2.2) for some $m_0 \in L^1(S)$ and the average $\frac{1}{N}\sum_{i=1}^{N}\varphi(\eta_i^{(N)})$ be bounded uniformly in N. Let m be a solution of the equation*

$$\frac{\partial m}{\partial t} = \frac{1}{2}\frac{\partial^2 h'(m)}{\partial \theta^2}, \quad (t, \theta) \in (0, \infty) \times S, \tag{2.17}$$

with the initial condition m_0. Then for any $t > 0$ and any test function $J \in C(S)$ the ν^t_{N,y_N}-measure of the set

$$U_{N,y_N}(J, \delta) = \Big\{x \in \pi_N(y_N) : \ \Big|\frac{1}{N}\sum_{i=1}^{N} J(\frac{i}{N})x_i(t) - \int_0^1 J(\theta)m(t, \theta)d\theta\Big| > \delta\Big\}$$

approaches zero as $N \to \infty$, i.e.

$$\lim_{N\to\infty} \int\limits_{U_{N,y_N}(J,\delta)} f^t_{N,y_N} d\mu_{N,y_N} = 0, \tag{2.18}$$

locally uniformly in $t > 0$.

2.3 Local Gibbs measures and main results

To every solution m of the hydrodynamic equation (2.17) we can associate a family of local Gibbs measures $\gamma_N^{m_t}$ on $\mathbb{R}^N$. Let $\lambda(t,\theta) = h'(m(t,\theta))$. Measures $\gamma_N^{m_t}$ are defined by their densities $g_N^{m_t}$ with respect to the invariant measure μ_{N,y_N}:

$$g_N^{m_t}(x) = \frac{d\gamma_N^{m_t}}{d\mu_N} = \frac{1}{Z_N^{m_t}} \exp\left(\sum_{i=1}^{N} \lambda\left(t, \frac{i}{N}\right) x_i\right)$$

where $Z_N^{m_t}$ is the normalization constant, $Z_N^{m_t} = \prod_{i\in\Lambda_N} M\left(\lambda\left(t, \frac{i}{N}\right)\right)$. Measures $\gamma_N^{m_t}$ depend only on the solution $m(t,\theta)$ and form a convenient set of reference measures. The conditional distribution of the local Gibbs measure $\gamma_N^{m_t}(\cdot\,|\sum_{i\in\Lambda_N} x_i^{(N)} = Ny)$ and its density relative to $\mu_{N,y}$ will be denoted by $\gamma_{N,y}^{m_t}$ and $g_{N,y}^{m_t}$ respectively. The main result of this chapter is the following theorem.

Theorem 2.2. *Let $\eta^{(N)}$, $N = 1, 2, \ldots,$ be a sequence of deterministic configurations which converges to an initial profile $m_0 \in L^1(S)$ as $N \to \infty$ in the sense of (2.2). Assume that φ satisfies the hypotheses $(H_1) - (H_3)$ and for some $C > 0$ and $\delta_0 > 0$ the following inequality holds:*

$$\frac{1}{N}\sum_{i=1}^{N} \varphi^{2(p+\delta_0)}(\eta_i^{(N)}) \leq C. \tag{2.19}$$

Denote the average $\frac{1}{N}\sum_{i=1}^{N} \eta_i^{(N)}$ by y_N. Then for any $t > 0$ the specific relative *entropy*

$$\frac{1}{N}H(\nu^t_{N,y_N}|\gamma^{m_t}_{N,y_N}) = \frac{1}{N}\int\limits_{\pi_N(y_N)} \log\frac{d\nu^t_{N,y_N}}{d\gamma^{m_t}_{N,y_N}}\,d\nu^t_{N,y_N}$$
$$= \frac{1}{N}\int\limits_{\pi_N(y_N)} f^t_{N,y_N}\,\log\frac{f^t_{N,y_N}}{g^{m_t}_{N,y_N}}\,d\mu_{N,y_N} \qquad (2.20)$$

approaches zero as $N \to \infty$.

We show also (see Lemma 2.1) that

$$\lim_{N\to\infty}\frac{1}{N}\int\limits_{\pi_N(y_N)} f^t_{N,y_N}\log g^{m_t}_{N,y_N}\,d\mu_{N,y_N} = \int_0^1 h(m(t,\theta))\,d\theta - h(\bar{m}),$$

where the constant $\bar{m}$ is defined by (2.3). Taking this into account the main result can be stated as follows:

Theorem 2.3. *Under assumptions of Theorem 2.2 for any fixed positive macroscopic time t the specific microscopic entropy converges as* $N \to \infty$ *to the macroscopic entropy, namely*

$$\lim_{N\to\infty} H_{N,y_N}(t) = \lim_{N\to\infty}\frac{1}{N}\int\limits_{\pi_N(y_N)} f^t_{N,y_N}\log f^t_{N,y_N}\,d\mu_{N,y_N}$$
$$= \int_0^1 h(m(t,\theta))\,d\theta - h(\bar{m})$$

2.4 Proof of Theorem 2.2

The basic steps of the proof are the same as for Theorem 1.2. But the realization of this program becomes more technical due to the fact that now our configuration space is non-compact.

Since

$$\frac{1}{N}H(\nu^t_{N,y_N}\,|\,\gamma^{m_t}_{N,y_N}) = \frac{1}{N}H(\nu^t_{N,y_N}\,|\,\mu_{N,y_N}) - \frac{1}{N}E_{\nu^t_{N,y_N}} \log g^{m_t}_{N,y_N}$$

and the relative entropy is non-negative, the following two lemmas imply the statement of Theorem 2.2.

Lemma 2.1. *Under conditions of Theorem 2.2*

$$\lim_{N\to\infty} \frac{1}{N} \int\limits_{\pi_N(y_N)} f^t_{N,y_N} \log g^{m_t}_{N,y_N}\, d\mu_{N,y_N} = \int_0^1 h(m(t,\theta))\,d\theta - h(\bar{m})$$

where $\bar{m}$ is defined by (2.3).

Lemma 2.2. *Under conditions of Theorem 2.2*

$$\limsup_{N\to\infty} \frac{1}{N} \int\limits_{\pi_N(y_N)} f^t_{N,y_N} \log f^t_{N,y_N}\, d\mu_{N,y_N} \le \int_0^1 h(m(t,\theta))\,d\theta - h(\bar{m}).$$

Proof of Lemma 2.1. Let $m(t,\theta)$ be the solution of (2.17) with initial condition m_0 and $\lambda(t,\theta) = h'(m(t,\theta))$. Denote $\lambda(\frac{i}{N},\theta)$ by λ^i_N. Recall that $y_N = \frac{1}{N}\sum_{i=1}^N \eta_i^{(N)}$. The density $g^{m_t}_{N,y_N}$ of a local Gibbs measure $\gamma^{m_t}_N$ with respect to the invariant measure μ_{N,y_N} is easily computed:

$$g^{m_t}_{N,y_N}(x) = \frac{\exp(\sum_{i=1}^N \lambda^i_N x_i) \int\limits_{\pi_N(y_N)} \exp(-\sum_{i=1}^N \varphi(x_i))\,d\sigma_N}{\int\limits_{\pi_N(y_N)} \exp(\sum_{i=1}^N \lambda^i_N x_i - \sum_{i=1}^N \varphi(x_i))\,d\sigma_N)}, \quad x \in \pi_N(y_N).$$

Hence

$$\frac{1}{N}\int\limits_{\pi_N(y_N)} \log g^{m_t}_{N,y_N}\, d\nu^t_{N,y_N} = \frac{1}{N}\int\limits_{\pi_N(y_N)} \sum_{i=1}^{N} \lambda^i_N x_i \, d\nu^t_{N,y_N} - \frac{1}{N}\log M(\lambda^i_N)$$

$$-\frac{1}{N}\log\Big(\prod_{i\in\Lambda_N} M(\lambda^i_N)\Big)^{-1} \int\limits_{\pi_N(y_N)} e^{\sum\limits_{i=1}^{N}\lambda^i_N x_i - \sum\limits_{i=1}^{N}\varphi(x_i)} d\sigma_N$$

$$+\frac{1}{N}\log \int\limits_{\pi_N(y_N)} e^{-\sum\limits_{i=1}^{N}\varphi(x_i)} d\sigma_N.$$

Evidently

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\log M(\lambda^i_N) = \int_0^1 \log M(\lambda(t,\theta))\, d\theta.$$

By Theorem A.2 and Theorem A.4 we obtain

$$\lim_{N\to\infty}\frac{1}{N}\log\int\limits_{\pi_N(y_N)} e^{-\sum\limits_{i=1}^{N}\varphi(x_i)} d\sigma_N = -h(\bar{m});$$

$$\lim_{N\to\infty}\frac{1}{N}\log\int\limits_{\pi_N(y_N)} e^{\sum\limits_{i=1}^{N}\lambda^i_N x_i - \sum\limits_{i=1}^{N}\varphi(x_i) - \sum\limits_{i=1}^{N}\log M(\lambda^i_N)} d\sigma_N = -I_\lambda(\bar{m}) = 0.$$

The last equality follows from (2.16), (A.14) and the definition of λ. If we show that

$$\lim_{N\to\infty} E_{\nu^t_{N,y_N}}\left|\frac{1}{N}\sum_{i=1}^{N}\lambda^i_N x_i - \int_0^1 \lambda(t,\theta)m(t,\theta)\, d\theta\right| = 0 \tag{2.21}$$

then the proof of Lemma 2.1 will be complete:

$$\lim_{N\to\infty}\frac{1}{N}\int\limits_{\pi_N(y_N)} \log g^{m_t}_{N,y_N}\, d\nu^t_{N,y_N} = \int_0^1 \big(\lambda m - \log M(\lambda)\big)(t,\theta)\, d\theta - h(\bar{m})$$

$$= \int_0^1 h(m(t,\theta))\, d\theta - h(\bar{m}).$$

To prove (2.21) it is enough to establish the uniform integrability of functions $\frac{1}{N}\sum_{i=1}^{N}|x_i|$ with respect to ν^t_{N,y_N} since by Theorem 2.18 we already know that for any $\delta > 0$

$$\lim_{N\to\infty} \nu^t_{N,y_N}\left(x \in \pi_N(y_N)\ :\ \left|\frac{1}{N}\sum_{i=1}^{N}\lambda^i_N x_i - \int_0^1 \lambda(t,\theta)m(t,\theta)\,d\theta\right| > \delta\right) = 0.$$

Proposition 2.1. *Let* $A_{N,R} = \{x \in \pi_N(y_N)\ :\ \frac{1}{N}\sum_{i=1}^{N}|x_i| > R\}$*. Then*

$$\lim_{R\to\infty}\int_{A_{N,R}} \frac{1}{N}\sum_{i=1}^{N}|x_i|\,d\nu^t_{N,y_N} = 0$$

uniformly in N *and* $t \in [t_0,\infty)$ *for any* $t_0 > 0$*.*

Proof. Under our assumptions on potential φ the proof is very easy (see the Remark below for more general case). Let $\omega(s) = \delta s^2$ where $\delta > 0$ will be chosen later. For any $\varepsilon > 0$ and all $R > (\delta\varepsilon)^{-1}$ we obtain

$$\begin{aligned}
\int_{A_{N,R}} \frac{1}{N}\sum_{i=1}^{N}|x_i|\,d\nu^t_{N,y_N} &\le \varepsilon\int_{A_{N,R}} \omega\Big(\tfrac{1}{N}\sum_{i=1}^{N}|x_i|\Big)\,d\nu^t_{N,y_N}\\
&\le \varepsilon\int_{A_{N,R}} \frac{1}{N}\sum_{i=1}^{N}\omega(|x_i|)\,d\nu^t_{N,y_N}\\
&\le \frac{\varepsilon}{N}\log\int\limits_{\pi_N(y_N)} e^{\sum_{i=1}^{N}\omega(|x_i|)}d\mu_{N,y_N} + \frac{\varepsilon}{N}\int\limits_{\pi_N(y_N)}\log\frac{d\nu^t_{N,y_N}}{d\mu_{N,y_N}}\,d\nu^t_{N,y_N}\\
&= \frac{\varepsilon}{N}\log\int\limits_{\pi_N(y_N)} e^{\sum_{i=1}^{N}\omega(|x_i|)-\sum_{i=1}^{N}\varphi(x_i)}d\sigma_N + \varepsilon H_{N,y_N}(t).
\end{aligned}$$

Since φ satisfies (H_2) we can choose $\delta = \frac{1}{2}C_0$ to ensure that

$$\int e^{\omega(|x|)-\varphi(x)}dx < \infty. \tag{2.22}$$

The statement now follows from Lemma A.2 and the entropy estimate (2.13).

Remark. Notice that in the proof of Proposition 2.1 we used only the fact that the function $\omega : [0,\infty) \to \mathbb{R}_+$ is convex, satisfies the condition (2.22), and $\lim_{N\to\infty} \frac{s}{\omega(s)} = 0$. But the existence of such function ω follows already from (H_2') (see, for example, [18]).Therefore, assuming (H_2'), the above proof depends only on the entropy bound (2.13). For the case of deterministic initial conditions (2.13) was proved in [11] for all φ which satisfy (H_2'), (H_3) and the inequalities

$$-B \le \varphi''(x) \le C_1\varphi(x) + C_2, \quad x \in \mathbb{R},$$

for some constants B, C_1 and C_2. The convexity of φ is not essential for the proof of Proposition 2.1. It is used later as a sufficient condition for the validity of the logarithmic Sobolev inequality.

Proof of Lemma 2.2. This proof depends on a number of auxiliary propositions. For convenience of the reader the precise statements are given in the appropriate steps below but the proofs are postponed till the next chapter.

We start with a martingale decomposition (see Proposition 1.1) repeating the construction given in the proof of Lemma 1.5. Let $\Omega = \pi_N(y_N)$ and $P = \mu_{N,y_N}$. Fix any $k \in \mathbb{N}$ and divide the circle S into k equal intervals (or "arcs") Δ_i, $i = 1, 2, \ldots, k$, of length $\varepsilon = \frac{1}{k}$. Without loss of generality we can assume that $N = kl$ where k is the number of "arcs" and $l = \varepsilon N$ is the number of sites in each "arc". Recall that the sites are numbered by the points of integer lattice Λ_N. The set of all site numbers corresponding to the i-th "arc" will be denoted by B_i, $B_i \subset \Lambda_N$.

Define

$$\bar{y}_i = \sum_{j=(i-1)l+1}^{il} x_j, \quad i = 1, 2, \dots, k;$$

$$\begin{aligned}\mathcal{G}_i = \{&\sigma\text{-algebra generated by coordinate functions } x_{il+1}, \dots, x_N \\ &\text{and by } \bar{y}_1, \dots, \bar{y}_i \text{ subject to the restriction} \\ &\textstyle\sum_{j=1}^{i} \bar{y}_i + \sum_{j=il+1}^{N} x_i = y_N\}, \quad i = 0, \dots, k.\end{aligned}$$

Fix an arbitrary $t > 0$ and let $f_i^t = E_{\mu_{N,y_N}} \left(f_{N,y_N}^t \mid \mathcal{G}_i \right), \; i = 1, \dots, k.$

By Proposition 1.1 we have

$$\begin{aligned}\frac{1}{N} H(\nu_{N,y_N}^t \mid \mu_{N,y_N}) &= \frac{1}{N} E_{\mu_{N,y_N}} f_{N,y_N}^t \log f_{N,y_N}^t \\ &= \frac{1}{N} \sum_{i=1}^{k} E_{\mu_{N,y_N}} E_{\mu_{N,y_N}} \left(f_{i-1}^t \log \frac{f_{i-1}^t}{f_i^t} \mid \mathcal{G}_i \right) \\ &+ \frac{1}{N} E_{\mu_{N,y_N}} f_k^t \log f_k^t. \qquad (2.23)\end{aligned}$$

Step 1. We apply the logarithmic Sobolev inequality stated below to the sum in the right hand side of (2.23).

Proposition 2.2. *Let* $\varphi \in C^2(\mathbb{R})$ *satisfies* (H_1) *and the convexity condition* $\varphi''(x) \geq C_0 > 0$ *for all* $x \in \mathbb{R}$. *Define* $\mu_{l,y}$ *to be the conditional of the product measure* $\exp\left(-\sum_{i=1}^{l} \varphi(x_i)\right) dx_1 dx_2 \dots dx_l$ *on the hyperplane* $\pi_l(y) = \{x \in \mathbb{R}^l \,:\, \sum_{i=1}^{l} x_i = ly\}$, *i.e.*

$$d\mu_{l,y} = \frac{1}{Z_{l,y}} e^{-\sum_{i=1}^{l} \varphi(x_i)} d\sigma_l,$$

where $Z_{l,y}$ *is a normalization constant for which* $E_{\mu_{l,y}} 1 = 1$ *and* $d\sigma_l$ *is the Lebesgue measure on* $\pi_l(y)$. *Then for any smooth* $f : \mathbb{R}^l \to \mathbb{R}_+$ *such that*

$E_{\mu_{l,y}} f = 1$

$$E_{\mu_{l,y}} f \log f \leq \frac{4l^2}{C_0} \widehat{D}_{l,y}(\sqrt{f}). \tag{2.24}$$

Here

$$\widehat{D}_{l,y}(g) = \frac{1}{2} \int\limits_{\pi_l(y)} \sum_{i=1}^{l-1} \left(\frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial x_{i+1}} \right)^2 d\mu_{n,y}.$$

Remark. This proposition is a consequence of the classical logarithmic Sobolev inequality [1]. See Chapter 3 for details.

Denote $\mu_{N,y_N}(\,\cdot\,|\mathcal{G}_i)$ by μ_i and the marginal of μ_i on the i-th "arc" by $\bar{\mu}_i$. It is easy to see that $\bar{\mu}_i$ is exactly of the form described in Proposition 2.2 (cf. $\mu_{l,\bar{y}_i}$). Setting $\bar{f}_i = f^t_{i-1}/f^t_i$ and applying (2.24) with $f = \bar{f}_i$ we obtain

$$\begin{aligned} E_{\bar{\mu}_i} \bar{f}_i \log \bar{f}_i &\leq Cl^2 \int\limits_{\pi_l(\bar{y}_i)} \sum_{j-1,j \in B_i} \left(\frac{\partial \sqrt{\bar{f}_i}}{\partial x_{j-1}} - \frac{\partial \sqrt{\bar{f}_i}}{\partial x_j} \right)^2 d\bar{\mu}_i \\ &= \frac{Cl^2}{4 f^t_i} \int\limits_{\pi_l(\bar{y}_i)} \sum_{j-1,j \in B_i} \frac{1}{f^t_{i-1}} \left(\frac{\partial f^t_{i-1}}{\partial x_{j-1}} - \frac{\partial f^t_{i-1}}{\partial x_j} \right)^2 d\bar{\mu}_i \end{aligned} \tag{2.25}$$

since f^t_i does not depend on coordinates in the i-th arc. Now we need to switch back from f^t_{i-1} to f^t_{N,y_N}. Let us denote $\left(\frac{\partial}{\partial x_{j-1}} - \frac{\partial}{\partial x_j} \right)$ by $\partial_{j-1,j}$. We have

$$\partial_{j-1,j} f^t_{i-1} = \partial_{j-1,j} E_{\mu_{i-1}} f^t_{N,y_N} = E_{\mu_{i-1}} \partial_{j-1,j} f^t_{N,y_N}. \tag{2.26}$$

The term where $\partial_{j-1,j}$ is applied to the density of μ_{i-1} vanishes because the dependence of the density on the parameters $x_j,\ j > (i-1)l$ is only through their sum $\sum_{n=(i-1)l+1}^{N} x_j$ and $\partial_{j-1,j}(\sum_{n=(i-1)l+1}^{N} x_n) = 0$. By (2.26) and Hölder inequality we can estimate

$$\int\limits_{\pi_l(\bar y_i)} \sum_{j-1,j\in B_i} \frac{\big(\partial_{j-1,j} f^t_{i-1}\big)^2}{f^t_{i-1}}\, d\bar\mu_i = \int\limits_{\pi_l(\bar y_i)} \sum_{j-1,j\in B_i} \frac{\big(E_{\mu_{i-1}} \partial_{j-1,j} f^t_{N,y_N}\big)^2}{f^t_{i-1}}\, d\bar\mu_i$$
$$\le \int\limits_{\pi_l(\bar y_i)} \sum_{j-1,j\in B_i} E_{\mu_{i-1}} \frac{(\partial_{j-1,j} f^t_{N,y_N})^2}{f^t_{N,y_N}}\, d\bar\mu_i$$
$$= E_{\mu_i} \sum_{j-1,j\in B_i} \frac{(\partial_{j-1,j} f^t_{N,y_N})^2}{f^t_{N,y_N}}. \tag{2.27}$$

Taking summation over i, averaging with respect to μ_{N,y_N} and substituting the result in (2.23) we conclude

$$\frac{1}{N} H(\nu^t_{N,y_N} \,|\, \mu_{N,y_N}) \le \frac{C\varepsilon^2}{N} D_{N,y_N}(\sqrt{f^t_{N,y_N}}) + \frac{1}{N} E_{\mu_{N,y_N}} f^t_k \log f^t_k$$

where C depends only on C_0 (see (H_3)).

To obtain a bound on the Dirichlét form we use (2.12) and (2.13). Assume that $\varepsilon \left(= \frac{1}{k}\right) < t$ and set $t_\varepsilon = t - \varepsilon$ then

$$\frac{1}{N} \int_{t_\varepsilon}^t D_{N,y_N}(\sqrt{f^s_{N,y_N}})\, ds = H_{N,y_N}(t_\varepsilon) - H_{N,y_N}(t) \le C(t_\varepsilon).$$

For any $\alpha > 1$ let $E^\varepsilon_{N,\alpha} = \{s \in (t_\varepsilon, t] \;:\; D_{N,y_N}(\sqrt{f^s_{N,y_N}}) \le \frac{\alpha N}{\varepsilon} C(t_\varepsilon\}$. Then the Lebesgue measure of $E^\varepsilon_{N,\alpha}$ is at least $(1 - \frac{1}{\alpha})\varepsilon$ and for any $t_N \in E^\varepsilon_{N,\alpha}$

$$H_{N,y_N}(t) \le H_{N,y_N}(t_N) \le \alpha\varepsilon C(t_\varepsilon, C_0) + \frac{1}{N} E_{\mu_{N,y_N}} f^{t_N}_k \log f^{t_N}_k. \tag{2.28}$$

In the next three steps we deal with the second term of (2.28). The function f^t_k depends only on the "average charges" $\bar y_1, \dots, \bar y_k$. Denote by ν^t_k the joint distribution of $\bar y_1, \dots, \bar y_k$ under ν^t_{N,y_N}. Let μ_k be the joint distribution of $\bar y_1, \dots, \bar y_k$ under the invariant measure μ_{N,y_N}. Thus $d\nu^t_k = f^t_k d\mu^t_k$ and

$$E_{\mu_{N,y_N}} f^{t_N}_k \log f^{t_N}_k = E_{\nu^{t_N}_{N,y_N}} \log \frac{d\nu^{t_N}_k}{d\mu_k}.$$

Step 2. For each N introduce a new process $\tilde{x}^{(N)}(t)$, which is the solution of the following system of stochastic differential equations:

$$d\tilde{x}(t) = N^2 A^{(N)} \tilde{b}^{(N)}(t, \tilde{x}(t))dt + NG^{(N)} d\beta^{(N)}$$

where $\beta^{(N)}$ is the standard N-dimensional Brownian motion,

$$A^{(N)} = G^{(N)} G^{(N)*}, \qquad \tilde{b}^{(N)}(t, \cdot) = \begin{cases} b^{(N)}(\cdot) & \text{if } t \leq \tilde{t}_N, \\ 0 & \text{if } t > \tilde{t}_N, \end{cases} \tag{2.29}$$

and $G^{(N)}$, $b^{(N)}(x)$ are given by (2.5). The value of $\tilde{t}_N \in (t_\varepsilon, t_N)$ will be chosen later. Assume that $\tilde{x}^{(N)}$ starts at $t = 0$ from the same deterministic condition $\eta^{(N)}$ as does $x^{(N)}$. In other words the new process has the same parameters as the original one up to the time $\tilde{t}_N$ after which the drift is "switched off" and from that moment on $\tilde{x}^{(N)}$ evolves as a pure Gaussian process on $\pi_N(y_N)$ with constant covariance.

The corresponding measure on the space of continuous paths will be denoted by $\tilde{P}_N$. For all quantities associated with the new process we keep the same notation as for $x^{(N)}$ but equip them with tilde.

Returning to the second term in (2.28) we can now split it into three parts and handle each part separately:

$$\frac{1}{N} E_{\nu^{t_N}_{N,y_N}} \log \frac{d\nu^{t_N}_k}{d\mu_k} = \frac{1}{N} E_{\nu^{t_N}_{N,y_N}} \log \frac{d\nu^{t_N}_k}{d\tilde{\nu}^{t_N}_k} + \\ \frac{1}{N} E_{\nu^{t_N}_{N,y_N}} \log \frac{d\tilde{\nu}^{t_N}_k}{d\sigma_k} - \frac{1}{N} E_{\nu^{t_N}_{N,y_N}} \log \frac{d\mu_k}{d\sigma_k}, \tag{2.30}$$

where $d\sigma_k$ is Lebesgue measure on $\pi_k(y_N) = \{x \in \mathbb{R}^k \,:\, \frac{1}{k} \sum_{i=1}^{k} x_i = y_N\}$.

From the convexity of the function $x \log x$ and Jensen's inequality for conditional expectations we have

$$\frac{1}{N} E_{\nu^t_{N,y_N}} \log \frac{d\nu_k^{t_N}}{d\tilde{\nu}_k^{t_N}} \leq \frac{1}{N} E_{P_N} \log \frac{dP_N}{d\tilde{P}_N}\bigg|_{\mathcal{F}_{t_N}} \tag{2.31}$$

where $\mathcal{F}_{t_N}$ is the σ-algebra up to time t_N. The right hand side of (2.31) can now be computed using Girsanov's formula:

Proposition 2.3.

$$\frac{1}{N} E_{P_N} \log \frac{dP_N}{d\tilde{P}_N}\bigg|_{\mathcal{F}_{t_N}} = \frac{N}{8} E_{P_N} \int_{\tilde{t}_N}^{t_N} \sum_{i \in \Lambda_N} (\varphi'(x_i(s)) - \varphi'(x_{i+1}(s)))^2 ds.$$

Applying Ito's formula to $\sum_{i \in \Lambda_N} \varphi(x_i(t))$ and taking into account that $x^{(N)}(t)$ satisfies (2.1) we can prove the following

Proposition 2.4. *Let* $\sum_{i \in \Lambda_N} \varphi(\eta_i^{(N)}) \leq CN$. *Then for any sufficiently large* α *there exist sequences* $\{t_N\}$ *and* $\{\tilde{t}_N\}$, $t_\varepsilon < \tilde{t}_N < t_N < t$, *such that*

i) $t_N \in E^{\varepsilon}_{N,\alpha}$;

ii) $E_{P_N} \int\limits_{\tilde{t}_N}^{t_N} \sum_{i \in \Lambda_N} (\varphi'(x_i(s)) - \varphi'(x_{i+1}(s)))^2 ds \leq CN(t_N - \tilde{t}_N)$;

iii) $t_N - \tilde{t}_N = cN^{-3}$ *for some positive constant* c.

Choose α large enough. Then from (2.31) and Propositions 2.3 and 2.4 we obtain

$$\frac{1}{N} E_{\nu^{t_N}_{N,y_N}} \log \frac{d\nu_k^{t_N}}{d\tilde{\nu}_k^{t_N}} \leq \frac{C}{N}. \tag{2.32}$$

Step 3. The second term in the right hand side of (2.30) can be computed explicitly starting from $\tilde{t}_N$ since the drift coefficient for $\tilde{x}^{(N)}$ vanishes for all $t > \tilde{t}_N$. After $\tilde{t}_N$ the "arc" averages $\tilde{y}_1, \ldots, \tilde{y}_k$ (defined by

$\tilde{y}_j = \frac{1}{l}\sum_{i\in\Delta_j}\tilde{x}_i^{(N)},\ j = 1,\ldots,k)$ of the process $x^{(N)}$ undergo a Gaussian diffusion in $\mathbb{R}^k$ with parameters $(0, k^2 A^{(k)})$ where

$$A^{(k)} = \begin{pmatrix} 2 & -1 & 0 & \ldots & -1 \\ -1 & 2 & -1 & \ldots & 0 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & 0 & \ldots & -1 \\ -1 & 0 & 0 & \ldots & 2 \end{pmatrix}$$

Let $\tilde{A}^{(k)}$ be the restriction of $A^{(k)}$ to $\pi_k(y_N) = \{y \in \mathbb{R}^k\ :\ \frac{1}{k}\sum_{i=1}^{k} y_i = y_N\}$. Then $\tilde{A}^{(k)}$ is non-degenerate and

$$\begin{aligned} \frac{1}{N}\log\frac{d\tilde{\nu}_k^{t_N}}{d\sigma_k}(\cdot) = \frac{1}{N}\log\Bigg[\Big((t_N-\tilde{t}_N)^{k-1}(2\pi k^2)^{k-1}\det\tilde{A}^{(k)}\Big)^{-1/2}\times \\ \int\limits_{\pi_k(y_N)}\exp\left(-\frac{1}{2(t_N-\tilde{t}_N)}(x-\cdot)^T\,\tilde{A}^{(k)\,-1}\,(x-\cdot)\right)d\tilde{\nu}_k^{\tilde{t}_N}\Bigg] \le \\ \frac{k-1}{2N}\big(\log(t_N-\tilde{t}_N)^{-1}-\log(2\pi)\big)-\frac{1}{2N}\log(k^{2(k-1)}\det\tilde{A}^{(k)}). \quad (2.33) \end{aligned}$$

The $\det\tilde{A}^{(k)}$ is equal to the product of non-zero eigenvalues of $A^{(k)}$.

Proposition 2.5. *We have*

$$\det\tilde{A}^{(k)} = \prod_{j=1}^{k-1}(1-\cos\frac{2\pi j}{k});$$

$$\lim_{k\to\infty}\frac{1}{k}\log(\det\tilde{A}^{(k)}) = \log 2 + \frac{4}{\pi}\int\limits_0^{\frac{\pi}{2}}\log\sin x\,dx.$$

From the above Proposition and *iii*) of Proposition 2.4 we see that for all k large enough

$$\frac{1}{N}\log\frac{d\tilde{\nu}_k^{t_N}}{d\sigma_k} \le \frac{k-1}{2N}\log(c^{-1}N^3) \le \frac{C\log N}{\varepsilon N}. \quad (2.34)$$

Step 4. Finally we show that

$$\lim_{k\to\infty} \liminf_{N\to\infty} \frac{1}{N} E_{\nu^{t_N}_{N,y_N}} \log \frac{d\mu_k}{d\sigma_k} = -\int_0^1 h(m(t,\theta)\, d\theta + h(\bar{m}). \tag{2.35}$$

Let

$$h_l(\bar{y}_j) \stackrel{\text{def}}{=} -\frac{1}{l} \log \int\limits_{\sum\limits_{j\in B_i} x_j = l\bar{y}_j} e^{-\sum\limits_{j\in B_i} \varphi(x_j)} d\sigma_N$$

Recall that $\bar{y}_i = \frac{1}{l}\sum_{j\in B_i} x_j$. By the definition of μ_k

$$\frac{1}{N} \log \frac{d\mu_k}{d\sigma_k} = -\frac{1}{k}\sum_{i=1}^{k} h_l(\bar{y}_j) - \frac{1}{N} \log \int\limits_{\pi_N(y_N)} e^{-\sum\limits_{i=1}^{N} \varphi(x_j)} d\sigma_N(x). \tag{2.36}$$

The second term on the right hand side of the above identity is a constant relative to $\nu^{t_N}_{N,y_N}$ and we know by (A.2) that

$$\lim_{N\to\infty} \frac{1}{N} \log \int\limits_{\pi_N(y_N)} e^{-\sum\limits_{i=1}^{N} \varphi(x_j)} d\sigma_N(x) = -h(\bar{m}).$$

We write the first sum in (2.36) as

$$-\frac{1}{k}\sum_{i=1}^{k} (h_l(\bar{y}_i) - h(\bar{y}_i)) - \frac{1}{k}\sum_{i=1}^{k} h(\bar{y}_i)$$

The following facts are already known:

a) $\lim_{l\to\infty} h_l(z) = h(z)$ locally uniformly in z (Theorem A.2);

b) $\lim\limits_{N\to\infty} \nu^{s}_{N,y_N}\left(x \in \mathbb{R}^N \,:\, |\bar{y}_i - k\int_{\Delta_i} m(s,\theta)\, d\theta| \geq \delta\right) = 0$ for any $\delta > 0$ locally uniformly in $s > 0$ (Theorem 2.1).

All we need to complete the proof is a uniform integrability estimate which would enable us to conclude that

$$\lim_{N\to\infty} E_{\nu^{t_N}_{N,y_N}} \frac{1}{k}\sum_{i=1}^{k} |h_l(\bar{y}_i) - h(\bar{y}_i)| = 0 \tag{2.37}$$

and also that for some $t_\varepsilon \in [t-\varepsilon, t]$

$$\liminf_{N\to\infty} E_{\nu^{t_N}_{N,y_N}} \left(-\frac{1}{k} \sum_{i=1}^{k} h(\bar{y}_i) \right) = -\frac{1}{k} \sum_{i=1}^{k} h(k \int_{\Delta_i} m(t_\varepsilon, \theta)\, d\theta). \tag{2.38}$$

As h and m are continuous functions and

$$k \int_{\Delta_i} m(t_\varepsilon, \theta)\, d\theta = m(t_\varepsilon, \theta_j) \quad \text{for some } \ \theta_i \in \Delta_i,$$

the statement (2.35) would follow by letting $k(=\frac{1}{\varepsilon})$ go to infinity.

Before formulating a sufficient condition for (2.37) and (2.38) to hold we investigate relationships between h_l, h and φ in more detail.

Proposition 2.6. *Let φ be smooth, satisfy (H_1), $\varphi''(x) \geq C_0$ for all large $|x|$ and $|\varphi'(x)| \leq C(\varphi(x)+1)$ for all $x \in \mathbb{R}$. Define*

$$h_l(x) = \frac{1}{l} \log \int\limits_{\sum_{j=1}^{l} x_i = lx} e^{-\sum_{j=1}^{l} \varphi(x_i)} d\sigma_l.$$

Then there is a constant $C > 0$ such that $|h_l(x)| \leq C(\varphi(x)+1)$, $x \in \mathbb{R}$, uniformly in l.

As an immediate consequence of this proposition we obtain the inequality $|h(x)| \leq C(\varphi(x)+1)$. In view of the above the following lemma gives us the desired uniform integrability.

Lemma 2.3. *Let all the conditions of Theorem 2.2 be satisfied and $q = \min\{1+\delta_0 p^{-1}, 2\}$. Then for any fixed k there exists $C > 0$ such that for any $i = 1, 2, \ldots, k$*

$$E_{\nu^s_{N,y_N}} |\varphi(\bar{y}_i)|^q \leq C$$

uniformly in N and locally uniformly in $s > 0$.

This finishes the proof of (2.35).

Conclusion. Using (2.32), (2.34) and (2.35) we can now estimate the right hand side of (2.30):

$$\limsup_{k\to\infty}\limsup_{N\to\infty}\frac{1}{N}E_{\nu^{t_N}_{N,y_N}}\log\frac{d\nu^{t_N}_k}{d\mu_k}\leq\int_0^1 h\left(m(t,\theta)\right)d\theta-h(\bar{m}).$$

This inequality together with (2.28) yields

$$\limsup_{N\to\infty}H_{N,y_N}(t)\leq\int_0^1 h\left(m(t,\theta)\right)d\theta-h(\bar{m})$$

which is precisely the statement of Lemma 2.2. □

The proof of Theorem 2.2 is now complete modulo the proofs of Propositions 2.2-2.6 and Lemma 2.3 which are the contents of the next chapter.

Chapter 3

Proofs of technical results for Ginzburg-Landau model

3.1 Proofs of Propositions 2.2 - 2.5

Proof of Proposition 2.2. This result is a simple consequence of

Theorem 3.1 ([1], see also [5]). *Let $U \in C^2(\mathbb{R}^n)$ be such that for some $\varepsilon > 0$ and all $x \in \mathbb{R}^n$ it satisfies Hess $U(x) \geq \varepsilon I$ and*

$$\int_{\mathbb{R}^n} \exp(-U(x))\, dx = 1.$$

Then for any smooth function $f\colon \mathbb{R}^n \to \mathbb{R}_+$ for which

$$\int_{\mathbb{R}^n} f(x) \exp(-U(x))\, dx = 1$$

the following inequality holds

$$\int_{\mathbb{R}^n} f(x) \log f(x) \exp(-U(x))\, dx \leq \frac{4}{\varepsilon} \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{f(x)} \exp(-U(x))\, dx. \quad (3.1)$$

Let $f\colon \mathbb{R}^n \to \mathbb{R}_+$ satisfy the assumptions of Proposition 2.2. Then

$$\int_{\pi_l(y)} f \log f \, d\mu_{l,y} = \frac{1}{Z_{l,y}} \int_{\mathbb{R}^{l-1}} f \log f(x_1, \ldots, x_l) e^{-\sum_{i=1}^{l} \varphi(x_i)} |J_{l-1}| \, d^{l-1}x$$

where $x_l = ly - \sum_{i=1}^{l-1} x_i$ and $J_{l-1} = \dfrac{d\sigma_l}{d^{l-1}x}$, $|J_{l-1}| = \sqrt{l-1}$. Set

$$U_y(x_1, \ldots, x_{l-1}) = \sum_{i=1}^{l-1} \varphi(x_i) + \varphi(ly - \sum_{i=1}^{l-1} x_i).$$

To apply (3.1) we need only to check that Hess $U_y \geq \varepsilon I$ for some $\varepsilon > 0$. By direct computation we obtain for any $z \in \mathbb{R}^{l-1}$

$$\langle (\text{Hess } U_y) z, z \rangle_{\mathbb{R}^{l-1}} = \sum_{i=1}^{l-1} \varphi(x_i) z_i^2 + \varphi''(x_l) \left(\sum_{i=1}^{l-1} x_i \right)^2 \geq C_0 \|z\|^2_{\mathbb{R}^{l-1}}.$$

From (3.1)we find that

$$\int_{\mathbb{R}^{l-1}} (f \log f) e^{-U_y} \, d^{l-1}x \leq \frac{4}{C_0} \int_{\mathbb{R}^{l-1}} \frac{1}{f} \sum_{i=1}^{l-1} \left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_l} \right)^2 e^{-U_y} \, d^{l-1}x.$$

The proof will be complete if we find an upper bound on the quadratic form $\sum_{i=1}^{l-1} (q_i - q_l)^2$ in terms of $\sum_{i=1}^{l-1} (q_i - q_{i+1})^2$ for all $q = (q_1, \ldots, q_l) \in \mathbb{R}^l$. We show that there are constants $0 < Q_1 < Q_2$ such that

$$Q_1 l^2 \leq \sup_{q \in \mathbb{R}^l \setminus 0} \frac{\sum_{i=1}^{l-1} (q_i - q_l)^2}{\sum_{i=1}^{l-1} (q_i - q_{i+1})^2} \leq Q_2 l^2$$

uniformly in $l \geq 2$. Indeed

$$\begin{aligned} \sum_{i=1}^{l-1} (q_i - q_l)^2 &= \sum_{i=1}^{l-1} \left(\sum_{j=i}^{l-1} (q_j - q_{j+1}) \right)^2 \leq \sum_{i=1}^{l-1} (l-i) \sum_{j=i}^{l-1} (q_j - q_{j+1})^2 \\ &\leq \left(\sum_{i=1}^{l-1} (l-i) \right) \sum_{i=1}^{l-1} (q_i - q_{i+1})^2 = \frac{l(l-1)}{2} \sum_{i=1}^{l-1} (q_i - q_{i+1})^2. \end{aligned}$$

This implies that $Q_2 \leq \frac{1}{2}$. Now if we take $q_i = i$ then

$$\frac{\sum_{i=1}^{l-1}(q_i - q_l)^2}{\sum_{i=1}^{l-1}(q_i - q_{i+1})^2} = \frac{\sum_{i=1}^{l-1} i^2}{l-1} = \frac{1}{3}(l-1)^2 + \frac{1}{2}(l-1) + \frac{1}{6} \geq \frac{1}{3}(l-1)^2.$$

The inequality $Q_1 > 0$ tells us that the order l^2 can not be improved. Thus

$$\sum_{i=1}^{l-1}\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_l}\right)^2 \leq \frac{l^2}{2}\sum_{i=1}^{l-1}\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{i+1}}\right)^2$$

and

$$\int_{\pi_l(y)} f \log f \, d\mu_{l,y} \leq \frac{2l^2}{C_0}\int_{\pi_l(y)} \frac{1}{f}\sum_{i=1}^{l-1}\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_l}\right)^2 d\mu_{l,y}$$

as claimed. □

Proof of Proposition 2.3. By Girsanov's formula

$$\log \frac{d\tilde{P}_N}{dP_N}\bigg|_{\mathcal{F}_{t_N}} =$$

$$-\int_0^{t_N} (b^{(N)} - \tilde{b}^{(N)})\, d\bar{x} - \frac{N^2}{2}\int_0^{t_N} \langle b^{(N)} - \tilde{b}^{(N)}, A^{(N)}(b^{(N)} - \tilde{b}^{(N)})\rangle\, ds$$

where $\bar{x} = x - N^2 \int_0^{t_N} A^{(N)} b^{(N)}\, ds$ is a P_N-martingale and $\langle \cdot, \cdot \rangle$ denotes a scalar product in $\mathbb{R}^N$. Since $\tilde{b}$ coincides with b up to time $\tilde{t}_N$ and vanishes for all $t > \tilde{t}_N$ we obtain

$$\log \frac{dP_N}{d\tilde{P}_N}\bigg|_{\mathcal{F}_{t_N}} = \int_{\tilde{t}_N}^{t_N} b^{(N)}\, d\bar{x} + \frac{N^2}{2}\int_{\tilde{t}_N}^{t_N} \langle b^{(N)}, A^{(N)} b^{(N)}\rangle\, ds$$

Taking expectation with respect to P_N in the above equality and using (2.5) we find that

$$\begin{aligned}\frac{1}{N} E_{P_N} \log \frac{dP_N}{d\tilde{P}_N}\bigg|_{\mathcal{F}_{t_N}} &= \frac{N}{2} E_{P_N} \int_{\tilde{t}_N}^{t_N} \langle b^{(N)}, A^{(N)} b^{(N)}\rangle\, ds \\ &= \frac{N}{8} E_{P_N} \int_{\tilde{t}_N}^{t_N} \sum_{i=1}^{N} \left(\varphi'(x_i) - \varphi'(x_{i+1})\right)^2\, ds.\end{aligned}$$

□

Proof of Proposition 2.4.

Step 1. We show that for any $t_0 > 0$ there is a constant $C = C(t_0)$ such that for all N and all $t > t_0$

$$\frac{1}{N} E_{\nu^t_{N,y_N}} \sum_{i=1}^{N} \varphi(x_i) \leq C(t_0). \tag{3.2}$$

By entropy inequality and (2.13) for any $\delta > 0$

$$\begin{aligned}\frac{1}{N} E_{\nu^t_{N,y_N}} \sum_{i=1}^{N} \varphi(x_i) &\leq \frac{1}{N\delta} \log E_{\mu_{N,y_N}} e^{\delta \sum_{i=1}^{N} \varphi(x_i)} + \frac{1}{N\delta} H_{N,y_N}(t) \\ &\leq \frac{1}{N\delta} \log E_{\mu_{N,y_N}} e^{\delta \sum_{i=1}^{N} \varphi(x_i)} + \frac{1}{\delta} C(t_0).\end{aligned}$$

Fix $\delta \in (0,1)$ then

$$\begin{aligned}\frac{1}{N} \log E_{\mu_{N,y_N}} e^{\delta \sum_{i=1}^{N} \varphi(x_i)} &= \frac{1}{N} \log \int_{\pi_N(y_N)} e^{-(1-\delta) \sum_{i=1}^{N} \varphi(x_i)} d\sigma_N \\ &\quad - \frac{1}{N} \log \int_{\pi_N(y_N)} e^{- \sum_{i=1}^{N} \varphi(x_i)} d\sigma_N \end{aligned} \tag{3.3}$$

By Theorem A.2 the right hand side of (3.3) has a finite limit as $N \to \infty$ which depends only on δ and $\bar{m}$. This proves the estimate (3.2).

Step 2. Using Itô's formula and (2.1) we show that for any $t_2 > t_1 > 0$

$$\frac{1}{2} E_{P_N} \int_{t_1}^{t_2} \sum_{i=1}^{N} \left(\varphi'(x_i(s)) - \varphi'(x_{i+1}(s))\right)^2 ds = \tag{3.4}$$

$$\frac{1}{N^2} \left(E_{P_N} \sum_{i=1}^{N} \varphi(x_i(t_1)) - E_{P_N} \sum_{i=1}^{N} \varphi(x_i(t_2)) \right) + E_{P_N} \int_{t_1}^{t_2} \sum_{i=1}^{N} \varphi''(x_i(s)) \, ds.$$

Indeed

$$d\varphi(x_i(s)) = \frac{N^2}{2}\varphi'(x_i(s))\left[\varphi'(x_{i-1}(s)) - 2\varphi'(x_i(s)) + \varphi(x_{i+1}(s))\right] ds$$
$$+ N^2\varphi''(x_i(s))\, ds + P_N\text{-martingale.}$$

Taking summation over i, integrating from t_1 to t_2 and computing the expectation with respect to P_N we find that

$$E_{P_N} \sum_{i\in\Lambda_N} \varphi(x_i(t_2)) =$$
$$E_{P_N} \sum_{i\in\Lambda_N} \varphi(x_i(t_1)) - \frac{N^2}{2} E_{P_N} \int_{t_1}^{t_2} \sum_{i\in\Lambda_N} \left(\varphi'(x_i(s)) - \varphi'(x_{i+1}(s))\right)^2 ds$$
$$+ N^2 E_{P_N} \int_{t_1}^{t_2} \sum_{i=1}^{N} \varphi''(x_i(s))\, ds$$

which implies (3.4).

Step 3. The first term in the right hand side of (3.4) will now be estimated. Take an arbitrary $h < \frac{\varepsilon}{2}$ ($\varepsilon = \frac{1}{k}$). Since $t > t_\varepsilon + h$ we obtain

$$\frac{1}{N}\int_{t_\varepsilon}^{t-h} \left(E_{P_N} \sum_{i=1}^{N} \varphi(x_i(s)) - E_{P_N} \sum_{i=1}^{N} \varphi(x_i(s+h)) \right) ds =$$
$$\frac{1}{N}\left(\int_{t_\varepsilon}^{t_\varepsilon+h} - \int_{t-h}^{t} \right) E_{P_N} \sum_{i=1}^{N} \varphi(x_i(s))\, ds \le 2hC(t_\varepsilon).$$

The last inequality follows from (3.2). This estimate implies that if we consider a set

$$\tilde{E}^{\varepsilon}_{N,\alpha} = \left\{ s \in [t_\varepsilon, t] \;:\; E_{P_N} \left(\sum_{i=1}^{N} \varphi(x_i(s)) - \sum_{i=1}^{N} \varphi(x_i(s+h)) \right) \le \alpha h N \right\}$$

where $\alpha > 0$ then its Lebesgue measure is not less than $(t - t_\varepsilon - h) - \frac{2}{\alpha}C(t_\varepsilon)$ and therefore is at least $\frac{\varepsilon}{2}(1 - \frac{4C(t_\varepsilon)}{\alpha\varepsilon})$. Consider now $\tilde{E}^{\varepsilon}_{N,\alpha} + \{h\}$ – a translation of $\tilde{E}^{\varepsilon}_{N,\alpha}$ by h. By choosing large enough α we can ensure that

$$\text{mes}[(\tilde{E}^{\varepsilon}_{N,\alpha} + \{h\}) \cap E^{\varepsilon}_{N,\alpha}] > \frac{\varepsilon}{4}.$$

Notice that α is independent of N and h.

The same reasoning goes through if we now let h depend on N so that $h(N) = h_N \to 0$ as $N \to \infty$. Choose any $t_N \in (\tilde{E}^{\varepsilon}_{N,\alpha} + \{h_N\}) \cap A^{\varepsilon}_{N,\alpha}$ and set $\tilde{t}_N = t_N - h_N$. Then $\tilde{t}_N \in \tilde{E}^{\varepsilon}_{N,\alpha}$ and for this choice of t_N and $\tilde{t}_N$ we have

$$\frac{1}{2} E_{P_N} \int_{\tilde{t}_N}^{t_N} \sum_{i=1}^{N} (\varphi'(x_i(s)) - \varphi'(x_{i+1}(s)))^2 \, ds \le \frac{\alpha h_N}{N} + E_{P_N} \int_{\tilde{t}_N}^{t_N} \sum_{i=1}^{N} \varphi''(x_i(s)) \, ds$$

Step 4. Since we assumed that $\varphi''(x) \le C_1(\varphi(x) + 1)$ (see (H_2)) the last term in the above inequality is bounded by

$$C_1 \Big(N(t_N - \tilde{t}_N) + E_{P_N} \int_{\tilde{t}_N}^{t_N} \sum_{i=1}^{N} \varphi(x_i(s)) \, ds \Big)$$

This implies that

$$\frac{1}{2} E_{P_N} \int_{\tilde{t}_N}^{t_N} \sum_{i=1}^{N} [\varphi'(x_i(s)) - \varphi'(x_{i+1}(s))]^2 \, ds \le \frac{1}{N} \alpha h_N + C_1 N h_N (1 + C(t_\varepsilon)).$$

The result now follows by setting $h_N = cN^{-3}$ for some $c > 0$. $\square$

Proof of Proposition 2.5. Denote $\frac{2\pi j}{k}$ by $\alpha_j^{(k)}$. It is easy to check that, since $-A^{(k)}$ is a matrix for a discrete Laplace operator with periodic boundary conditions, the vectors $e_j^{(k)} = (e^{i\alpha_j^{(k)}}, \dots, e^{ik\alpha_j^{(k)}})$, $j = 0, \dots, k$, are independent eigenvectors for $A^{(k)}$ with corresponding eigenvalues given by $(1 - \cos \alpha_j^{(k)})$. Taking the product of non-zero eigenvalues ($j = 1, \dots, k-1$) we obtain the first part of the proposition.

To prove the second statement notice that

$$\frac{1}{k}\log\prod_{j=1}^{k-1}(1-\cos\frac{2\pi j}{k}) = \frac{k-1}{k}\log 2 + \sum_{i=1}^{k-1}\frac{1}{k}\log\sin^2\frac{\pi j}{k}$$

$$= \frac{k-1}{k}\log 2 + 4\sum_{j=1}^{\left[\frac{k}{2}\right]}\frac{1}{k}\log\sin\frac{\pi j}{k}. \qquad (3.5)$$

The last sum is an integral sum for the convergent integral

$$\int_0^{\frac{1}{2}}\log\sin(\pi x)\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\log\sin x\,dx.$$

Passing to the limit as $k\to\infty$ in (3.5) yields the desired result. □

3.2 Proofs of Proposition 2.6 and Lemma 2.3

Proof of Proposition 2.6. For any $x' < x''$ it is true that

$$\int_{x'}^{x''} e^{-h_l(x)}dx = \int_{lx'\le\sum_{j=1}^{l}x_j\le lx''} e^{-\sum_{j=1}^{l}\varphi(x_j)}\,d\sigma_l$$

$$\geq \int_{\substack{x'\le x_j\le x''\\ j=1,\dots,l}} e^{-\sum_{j=1}^{l}\varphi(x_j)}\,d^lx = \left(\int_{x'}^{x''}e^{-\varphi(x)}dx\right)^l$$

If h_l and φ are monotone increasing for $x > x_0 > 0$ then

$$(x''-x')\exp(-lh_l(x')) \geq (x''-x')^l\exp(-l\varphi(x'')) \quad \text{for } x', x''\in[x_0,\infty),$$

i.e $h_l(x') \leq -(1-\frac{1}{l})\log(x''-x') + \varphi(x'')$ which implies

$$h_l(x) \leq \varphi(x+1) \qquad \text{for } x > x_0. \qquad (3.6)$$

Similarly if h_l and φ are monotone decreasing for $x < -x_0$ then

$$h_l(x) \leq \varphi(x-1) \qquad \text{for } x < -x_0. \tag{3.7}$$

Since φ is convex and $\int e^{-\varphi(x)}dx < \infty$ we know that it is monotone increasing for large positive x and is monotone decreasing for large negative x. Moreover by our assumptions (see (H_2))

$$\varphi'(x) \geq C_0 x - C \qquad \text{for } x > x_0 \quad \text{and}$$
$$\varphi'(x) \leq C_0 x + C \qquad \text{for } x < -x_0$$

for some $x_0 > 0$. This inequalities hold also for h_l':

$$h_l'(x) = \frac{\int_{\sum_{j=1}^{l} x_j = lx} \frac{1}{l}\sum_{j=1}^{l}\varphi'(x_j) e^{-\sum_{j=1}^{l}\varphi(x_j)} d\sigma_l}{\int_{\sum_{j=1}^{l} x_j = lx} e^{-\sum_{j=1}^{l}\varphi(x_j)} d\sigma_l} \geq C_0 x - C \quad \text{for } x > x_0$$

and similarly $h_l'(x) \leq C_0 x + C$ for $x < -x_0$. This implies monotonicity of h_l for large $|x|$. Using the condition on the derivative of φ (see (H_3)) we find that

$$\varphi(x \pm 1) \leq C(\varphi(x) + 1) \tag{3.8}$$

for some constant $C > 0$. The statement now follows from (3.6) - (3.8) by noticing that h_l are bounded below uniformly in l. The last observation is a consequence of monotonicity of h_l and Theorem A.2. □

Proof of Lemma 2.3. Without loss of generality we can assume that $\varphi \geq 0$.

Step 1. We have

$$E_{\nu^s_{N,y_N}} \varphi^q(\bar{y}_i) = E_{\nu^s_{N,y_N}} \varphi^q\left(\frac{1}{l}\sum_{j\in B_i} x_j\right) \le \frac{1}{l^q} E_{\nu^s_{N,y_N}} \left(\sum_{j\in B_i} \varphi(x_j)\right)^q.$$

Let

$$H^{(q)}_{N,y_N}(s) = \frac{1}{N^q} \int\limits_{\pi_N(y_N)} f^s_{N,y_N} \left|\log f^s_{N,y_N}\right|^q \, d\mu_{N,y_N}.$$

By Lemma B.1 for any $\delta > 0$ we have

$$\frac{1}{N^q} E_{\nu^s_{N,y_N}} \left(\sum_{j\in B_i} \varphi(x_j)\right)^q \le \frac{2}{(\delta N)^q} \log^q 3 \left(1 + E_{\mu_{N,y_N}} e^{\delta \sum\limits_{j\in B_i} \varphi(x_j)}\right) + \frac{6}{\delta^q} H^{(q)}_{N,y_N}(s). \tag{3.9}$$

The first term on the right hand side of (3.9) is clearly bounded since

$$\chi_N = \frac{1}{N} \log E_{\mu_{N,y_N}} e^{\delta \sum\limits_{j\in B_i} \varphi(x_j)}$$

converges to a limit as $N \to \infty$ (see (3.3)) and

$$0 \le \frac{1}{N} \log(1 + \chi_N) \le \begin{cases} \frac{1}{N} \log 2 & \text{if } \chi_N \le 1 \\ \frac{1}{N} \log(2\chi_N) & \text{if } \chi_N > 1. \end{cases}$$

Thus we have reduced the problem to proving that for our choice of q

$$H^{(q)}_{N,y_N}(s) \le C_q \tag{3.10}$$

locally uniformly in $s > 0$.

The proof of (3.10) is similar to the proof of (2.13) given in [11]. Even though the function $x|\log x|^q$ is convex only for $x \ge \exp(1-q)$ it will not affect our considerations. There exists a smooth convex function $K(x)$ such that

$$x|\log x|^q \le K(x) \le B + x|\log x|^q, \quad x \ge 0 \tag{3.11}$$

for some $B > O$. We have

$$\frac{d}{ds} \int\limits_{\pi_N(y_N)} K(f^s_{N,y_N})\, d\mu_{N,y_N} = \int\limits_{\pi_N(y_N)} K'(f^s_{N,y_N}) \mathcal{L}_N f^s_{N,y_N}\, d\mu_{N,y_N}$$

$$= - \int\limits_{\pi_N(y_N)} K''(f^s_{N,y_N}) \sum_{i \in \Lambda_N} \left[\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right) f^s_{N,y_N} \right]^2 d\mu_{N,y_N} \le 0$$

and therefore if (3.10) holds for some $s_0 > 0$ with $C_q = C_q(s_0)$ then it holds for all $s \ge s_0$ with $C_q \le C_q(s_0) + B$. In the next step we argue that if we let s_0 depend on N in such a way that $s_0(N) \to 0$ as $N \to \infty$ and $s_0(N)N^2$ is bounded below by some $\tau > 0$ then the sequence $H^{(q)}_{N,y_N}(s_0(N))$ remains bounded by a constant $C = C(\tau)$. This will finish the proof of (3.10).

Step 2. Let $s_N = \tau N^{-2}$, $\tau > 0$, and define a non-speeded process $z^{(N)}(\tau) = x^{(N)}(s_N)$. Then $z^{(N)}(\tau)$ satisfies the system

$$dz_i(\tau) = \frac{1}{2} \left(\varphi'(z_{i-1}) - 2\varphi(z_i) + \varphi(z_{i+1}) \right) d\tau + d\beta_i(\tau) - d\beta_{i+1}(\tau), \qquad (3.12)$$

with initial conditions $z_i(0) = \eta_i^{(N)}$, $i \in \Lambda_N$. By abuse of notation we write f^τ_{N,y_N}, ν^τ_{N,n_N}, $H^{(q)}_{N,y_N}(\tau)$ instead of $f^{s_N}_{N,y_N}$, $\nu^{s_N}_{N,y_N}$, $H^{(q)}_{N,y_N}(s_N)$.

We need to show that

$$H^{(q)}_{N,y_N}(\tau) = \frac{1}{N^q} \int\limits_{\pi_N(y_N)} f^\tau_{N,y_N} |\log f^\tau_{N,y_N}|^q\, d\mu_{N,y_N} \le C(\tau)$$

uniformly in N. This is done by "comparison" of $z^{(N)}$ with a Gaussian process $w^{(N)}$ which solves

$$dw_i^{(N)}(\tau) = d\beta_i^{(N)}(\tau) - d\beta_{i+1}^{(N)}(\tau), \qquad i \in \Lambda_N,$$

with the same initial data.

Denote by r^{τ}_{N,y_N} the density of the finite dimensional distribution of $w^{(N)}(\tau)$ with respect to μ_{N,y_N}. Then

$$\int\limits_{\pi_N(y_N)} f^{\tau}_{N,y_N} |\log f^{\tau}_{N,y_N}|^q \, d\mu_{N,y_N} \leq$$

$$\leq \int\limits_{\{f^{\tau}_{N,y_N} \geq 1\}} f^{\tau}_{N,y_N} \left| \log \left(\frac{f^{\tau}_{N,y_N}}{r^{\tau}_{N,y_N}} \cdot r^{\tau}_{N,y_N} \right) \right|^q d\mu_{N,y_N} + \max_{0 \leq x \leq 1} x |\log x|^q.$$

Since for any $0 < x \leq y, \ xy \geq 1$ it is true that

$$0 \leq \log(xy) \leq 2 \log y$$

we have

$$\int\limits_{\{f^{\tau}_{N,y_N} \geq 1\}} f^{\tau}_{N,y_N} \left| \log \left(\frac{f^{\tau}_{N,y_N}}{r^{\tau}_{N,y_N}} \cdot r^{\tau}_{N,y_N} \right) \right|^q d\mu_{N,y_N} \leq$$

$$\leq 2^q \int\limits_{\{f^{\tau}_{N,y_N} \geq 1\}} f^{\tau}_{N,y_N} \left(\max \left\{ \log \frac{f^{\tau}_{N,y_N}}{r^{\tau}_{N,y_N}}, \log r^{\tau}_{N,y_N} \right\} \right)^q d\mu_{N,y_N}$$

$$\leq 2^q \Bigg(\int\limits_{\pi_N(y_N)} f^{\tau}_{N,y_N} \left| \log \frac{f^{\tau}_{N,y_N}}{r^{\tau}_{N,y_N}} \right|^q d\mu_{N,y_N} + \int\limits_{\{r^{\tau}_{N,y_N} \geq 1\}} f^{\tau}_{N,y_N} \log^q r^{\tau}_{N,y_N} \, d\mu_{N,y_N} \Bigg).$$

The first term of the sum above will be estimated by Girsanov's formula. At this point our assumption (2.19) on the initial data comes into play. The fact that r^{τ}_{N,y_N} is known explicitly will allow us to obtain a bound on the second term. The following two propositions complete the proof of Lemma 2.3.

Proposition 3.1. *Assume that $\{\eta^{(N)}\}$ satisfies (2.19) where p is defined in (H_3). Then*

i) $E_{\nu^{\tau}_{N,n_N}} \sum_{i \in \Lambda_N} |\varphi(z_i^{(N)})|^{2pq} \leq C(T)N$ *for all* $\tau \in [0,T], \ T < \infty$*;*

ii) $\int\limits_{\pi_N(y_N)} f^\tau_{N,y_N} \left| \log \frac{f^\tau_{N,y_N}}{r^\tau_{N,y_N}} \right|^q d\mu_{N,y_N} \le C(\tau) N^q$ *for any* $\tau > 0$.

Proposition 3.2. *For any* $\tau > 0$ *there is a constant* $C(\tau)$ *such that*

$$\int\limits_{r^\tau_{N,y_N} \ge 1} f^\tau_{N,y_N} \log^q r^\tau_{N,y_N} \, d\mu_{N,y_N} \le C(\tau) N^q.$$

Proof of Proposition 3.1. To simplify the notation we drop the superscript in $z^{(N)}$.

i) First we prove a slightly more general statement. Let $\psi(x)$ be any smooth convex function which satisfies the inequality

$$\psi''(x) \le C(\psi(x) + 1) \tag{3.13}$$

Then by Itô's formula

$$d\psi(z_i) = \psi''(z_i) d\tau + \psi'(z_i) dz_i.$$

Taking summation over i, integrating from 0 to τ, applying (3.12) and computing the expectation we get

$$\begin{aligned} E_{P_N} \sum_{i \in \Lambda_N} \psi(z_i(\tau)) &= \sum_{i \in \Lambda_N} \psi(\eta_i^{(N)}) + \int\limits_0^\tau E_{P_N} \psi''(z_i(s)) \, ds \\ &- \frac{1}{2} \int\limits_0^\tau E_{P_N} \sum_{i \in \Lambda_N} \big(\psi'(z_{i+1}(s)) - \psi'(z_i(s))\big)\big(\varphi'(z_{i+1}(s)) - \varphi'(z_i(s))\big) \, ds \\ &\le \sum_{i \in \Lambda_N} \psi(\eta_i^{(N)}) + CN\tau + C \int\limits_0^\tau E_{P_N} \psi(z_i(s)) \, ds. \end{aligned}$$

In the last inequality we used convexity of φ and ψ and the condition (3.13). An application of the Gronwall inequality yields

$$E_{\nu^\tau_{N,n_N}} \sum_{i \in \Lambda_N} \psi(z_i) \le \left(\sum_{i \in \Lambda_N} \psi(\eta_i^{(N)}) + N(1 - e^{-C\tau}) \right) e^{C\tau}. \tag{3.14}$$

In our case we can take $\psi(x)$ essentially to be equal to $|\varphi(x)|^{2pq}$ modifying the latter on a finite interval as necessary. Then from (3.14) we obtain for $\tau \in [0,T]$

$$E_{\nu^{\tau}_{N,n_N}} \sum_{i\in\Lambda_N} |\varphi(z_i)|^{2pq} \le C(T)N.$$

ii) We show that $\int\limits_{\pi_N(y_N)} K\left(\frac{f^{\tau}_{N,y_N}}{r^{\tau}_{N,y_N}}\right) r^{\tau}_{N,y_N}\, d\mu_{N,y_N} \le C(\tau)N^q$ where K satisfies (3.11). This immediately implies *ii*). By convexity of K

$$\int\limits_{\pi_N(y_N)} K\left(\frac{f^{\tau}_{N,y_N}}{r^{\tau}_{N,y_N}}\right) r^{\tau}_{N,y_N}\, d\mu_{N,y_N} \le E_{Q_N} K\left(\frac{dP_N}{dQ_N}\right)_{\mathcal{F}_\tau}\Bigg|$$

$$\le B + E_{P_N}\big|\log \frac{dP_N}{dQ_N}\Big|_{\mathcal{F}_\tau}\big|^q.$$

where Q_N is the measure on continuous paths associated with $w^{(N)}$. In the same way as in the proof of Proposition 2.3 we obtain

$$E_{P_N}\big|\log \frac{dP_N}{dQ_N}\Big|_{\mathcal{F}_\tau}\big|^q =$$

$$= E_{P_N}\left|\int\limits_0^\tau b^{(N)}(z(s))\, d\bar z(s) + \frac{1}{2}\int_0^\tau \langle b^{(N)}(z), A^{(N)} b^{(N)}(z)\rangle(s)\, ds\right|^q$$

$$\le 2^{q-1}\left|E_{P_N}\int\limits_0^\tau b^{(N)}(z(s))\, d\bar z(s)\right|^q + C(\tau)E_{P_N}\int\limits_0^\tau \langle b^{(N)}(z), A^{(N)} b^{(N)}(z)\rangle^q(s)\, ds$$

with $\bar z(s) = z(s) - \int\limits_0^\tau A^{(N)} b^{(N)}(z(s))\, ds.$ From the inequality

$a^q \leq \dfrac{2}{q}\, a^2 + \left(\dfrac{2}{q} - 1\right)$, $q \in [1,2]$, and Itô's isometry we find that

$$E_{P_N}\left|\log \frac{dP_N}{dQ_N}\right|_{\mathcal{F}_\tau}|^q \leq \frac{2^q}{q} E_{P_N} \int\limits_0^\tau \langle b^{(N)}(z), A^{(N)} b^{(N)}(z)\rangle(s)\, ds + 2^{q-1}\left(\frac{2}{q} - 1\right)$$

$$+ C(\tau) E_{P_N} \int\limits_0^\tau \langle b^{(N)}(z(s)), A^{(N)} b^{(N)}(z(s))\rangle^q\, ds$$

$$< \tilde{C}(\tau)\big(E_{P_N} \int\limits_0^\tau \langle b^{(N)}(z), A^{(N)} b^{(N)}(z)\rangle^q(s)\, ds + 1\big).$$

Notice that by (2.5) and (H_3)

$$\langle b^{(N)}(z), A(N) b^{(N)}(z)\rangle = \frac{1}{2} \sum_{i \in \Lambda_N} \big(\varphi'(z_{i+1}) - \varphi'(z_i)\big)^2$$

$$\leq 4C_1^2\big(\sum_{i \in \Lambda_N} |\varphi(z_i)|^{2p} + N\big).$$

Therefore by the Hölder inequality and part $i)$

$$E_{P_N} \int\limits_0^\tau \langle b^{(N)}(z(s)), A(N) b^{(N)}(z(s))\rangle^q\, ds$$

$$\leq C\left(N^{q-1} E_{P_N} \int\limits_0^\tau \sum_{i \in \Lambda_N} |\varphi(z_i(s))|^{2pq}\, ds + N^q\right) \leq C(\tau) N^q$$

This completes the proof. □

Proof of Proposition 3.2. Let $\tilde{A}^{(N)}$ be the restriction of $A^{(N)}$ to $\pi_N(y_N)$. Then by the definition of r^τ_{N,y_N} for any z such that $r^\tau_{N,y_N}(z) \geq 1$ we obtain

$$
\begin{aligned}
&0 \le \frac{1}{N}\log r^{\tau}_{N,y_N}(z) = \\
&\frac{1}{N}\log\Big[\Big((2\pi\tau)^{N-1}\det\tilde{A}^{(N)}\Big)^{-\frac{1}{2}}\exp\Big(-\frac{1}{2\tau}(z-\eta^{(N)})^T\ \tilde{A}^{(N)\,-1}\,(z-\eta^{(N)})\Big)\Big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \frac{1}{N}\log\frac{d\mu_{N,y_N}}{d\sigma_N}(z) \\
&\le -\frac{N-1}{2N}\log(2\pi\tau) - \frac{1}{2N}\log(\det\tilde{A}^{(N)}) \\
&\qquad\qquad\qquad + \frac{1}{N}\sum_{i=1}^{N}\varphi(z_i) + \frac{1}{N}\log\int\limits_{\pi_N(y_N)} e^{-\sum\limits_{i=1}^{N}\varphi(x_i)}d\sigma_N
\end{aligned}
$$

The first term in the right hand side is evidently bounded. The second and the fourth terms have finite limits as $N\to\infty$ by Proposition 2.5 (with $k=N$) and Theorem A.2 respectively. The third term is the only one which depends on z. We conclude that

$$
\begin{aligned}
\frac{1}{N^q}\int\limits_{r^{\tau}_{N,y_N}\ge 1} f^{\tau}_{N,y_N}\log^q r^{\tau}_{N,y_N}\,d\mu_{N,y_N} &\le C(\tau) + \frac{2^{q-1}}{N^q}\int\limits_{\pi_N(y_N)} \Big|\sum_{i\in\Lambda_N}\varphi(z_i)\Big|^q\,d\nu^{\tau}_{N,n_N} \\
&\le C(\tau) + \frac{2^{q-1}}{N^q}\int\limits_{\pi_N(y_N)} N^{q-\frac{1}{2p}}\Big(\sum_{i\in\Lambda_N}|\varphi(z_i)|^{2pq}\Big)^{\frac{1}{2p}}d\nu^{\tau}_{N,n_N} \\
&= C(\tau) + 2^{q-1}\bigg(E_{\nu^{\tau}_{N,n_N}}\frac{1}{N}\sum_{i\in\Lambda_N}|\varphi(z_i)|^{2pq}\bigg)^{\frac{1}{2p}} \le \tilde{C}(\tau)
\end{aligned}
$$

by part $i)$ of Proposition 3.1. □

Appendix A

Limit theorems for densities

This appendix contains limit theorems concerning probability densities for sums of independent but not identically distributed random variables. More precisely, let $f(x,\gamma) : \mathbb{X} \to \mathbb{R}$ be a probability distribution with the state space $\mathbb{X}$ for each $\gamma \in \mathbb{R}$ and be a continuous function of γ. For any continuous function λ on $[0,1]$ denote by λ_N^i its value at the point $\frac{i}{N}$, $i = 1, 2, .., N$. We consider independent random variables X_N^i, $N = 1, 2, \ldots$, $i = 1, \ldots, N$ such that the distribution of each X_N^i is given by $f(\cdot, \lambda_N^i)$. We prove large deviation theorems for Bernoulli case and some continuous densities.

A.1 Discrete case

Let λ be as above. Define

$$m(\theta) = \frac{e^{\lambda(\theta)}}{1 + e^{\lambda(\theta)}},\ \theta \in [0,1],\quad m_N^i = m(\tfrac{i}{N}),\ i = 1, \ldots, N. \tag{A.1}$$

Then $m \in C([0,1];(0,1))$. Let X_N^i, $N \in \mathbb{N}$, $i = 1, \ldots, N$, be independent random variables for which $P\{X_N^i = 1\} = m_N^i$, $P\{X_N^i = 0\} = 1 - m_N^i$.

Theorem A.1. *Let n_N, $N \in \mathbb{N}$, be an integer between 0 and N inclusively and $\lim\limits_{N\to\infty} \frac{n_N}{N} = y$ for some $y \in [0,1]$. Consider*

$$P_N(n_N) \stackrel{\text{def}}{=} P\Big\{\sum_{i=1}^{N} X_N^i = n_N\Big\}.$$

Then

$$\lim_{N\to\infty} \frac{1}{N} \log P_N(n_N) = -R_\lambda(y)$$

where

$$R_\lambda(y) = \sup_{z\in\mathbb{R}} \left(yz - \int_0^1 \rho(\lambda(\theta)+z)\,d\theta + \int_0^1 \rho(\lambda(\theta))\,d\theta \right),$$

$$\rho(\lambda) = \log(1+e^\lambda).$$

Remark. If $y = \bar{m} = \int_0^1 m(\theta)\,d\theta$ then $R_\lambda(\bar{m}) = 0$.

Proof. Let ψ_N^i be the characteristic function of X_N^i. Then

$$\psi_N^i(\xi) = e^{i\xi} m_N^i + (1 - m_N^i)$$

and

$$P_N(n_N) = \frac{1}{2\pi} \int_0^{2\pi} e^{in_N\xi} \prod_{i\in\Lambda_N} \psi_N^i(\xi)\,d\xi.$$

Denote the function under the integral sign by $F_N(\xi)$. It is periodic with period 2π, analytic and has N zeroes on the line $\operatorname{Re}\xi = \pi$. For $\xi \in \{-\pi < \operatorname{Re}\xi < \pi\}$ we can represent $F_N(\xi)$ as $e^{NS_{N,y_N}(\xi)}$ where $y_N = \frac{n_N}{N}$ and

$$\begin{aligned} S_{N,y_N}(\xi) &= -i\frac{n_N}{N}\xi + \frac{1}{N}\sum_{i=1}^{N} \left(e^{i\xi} m_N^i + (1-m_N^i)\right) \\ &= iy_N\xi + \frac{1}{N}\sum_{i=1}^{N} \rho(i\xi + \lambda_N^i) - \frac{1}{N}\sum_{i=1}^{N}\rho(\lambda_N^i) \end{aligned} \tag{A.2}$$

is an analytic function which converges as $N \to \infty$ to

$$S_{\lambda,y}(\xi) = -iy\xi + \int_0^1 \rho(i\xi + \lambda(\theta))\, d\theta - \int_0^1 \rho(\lambda(\theta))\, d\theta$$

locally uniformly in $\xi \in \{-\pi < \operatorname{Re} \xi < \pi\}$.

We apply the saddle-point method and show that the main contribution to $P_N(n_N)$ comes from a small neighborhood of a single critical point of S_{N,y_N} away from the line $\operatorname{Re} \xi = \pi$. On that line the function $F_N(\xi)$ may have zeroes and thus $S_{N,y_N} = \frac{1}{N} \log F_N$ may not be well defined.

The critical points of S_{N,y_N} are the roots of the equation

$$y_N = \frac{1}{N} \sum_{i=1}^{N} \rho'(i\xi + \lambda_N^i) = \frac{1}{N} \sum_{i=1}^{N} \frac{e^{i\xi + \lambda_N^i}}{1 + e^{i\xi + \lambda_N^i}}.$$

Since y_N is real, the imaginary part of the right hand side should be zero for any root ξ_0^N of the above equation. This implies that $\operatorname{Re} \xi_0^N = 0$ (we consider only $\xi \in \{-\pi < \operatorname{Re} \xi \leq \pi\}$). But for purely imaginary ξ, $\xi = i\beta$, the function $\rho'(i\xi + \lambda_N^i) = \rho'(\lambda_N^i - \beta)$ is monotone decreasing from 1 to 0 as β increases from $-\infty$ to $+\infty$ and therefore for any $y_N \in (0, 1)$ the above equation has a single root $\xi_0^N = i\beta_0^N$, $\beta_0^N \in \mathbb{R}$. This is certainly the case when $y \in (0, 1)$ and N is large. If $y = 0$ or $y = 1$ then the critical point ξ_0^N might be at $+i\infty$ or $-i\infty$ respectively. But since the limit of $S_{N,y_N}(i\beta)$ as $\beta \to \pm\infty$, $\beta \in \mathbb{R}$, is finite the proof for these cases goes along the same lines as for $y \in (0, 1)$ and is omitted. We assume now that $y \in (0, 1)$.

It is easy to check that the restriction of the real part of S_{N,y_N} to the imaginary axis attains its absolute minimum at ξ_0^N and the restriction to any interval $\{i\beta + t,\ t \in (-\pi, \pi)\}$, where $\beta \in \mathbb{R}$ is fixed, attains its absolute maximum at $i\beta$. Let $\delta_N = \delta N^{-\varepsilon}$ where $\delta > 0$ is small and $\varepsilon \in \left(\frac{1}{3}, \frac{1}{2}\right)$. We have

$$P_N(n_N) = \frac{1}{2\pi}\int\limits_{\xi_0^N-\delta_N}^{\xi_0^N+\delta_N} e^{NS_{N,y_N}}\,d\xi + \frac{1}{2\pi}\int\limits_{\xi_0^N+\delta_N}^{\xi_0^N+2\pi-\delta_N} F_N(\xi)\,d\xi = I_1 + I_2.$$

It is a standard computation (see, for example, [2] p.92-93) that as $N \to \infty$

$$I_1 = \frac{e^{NS_{N,y_N}(\xi_0^N)}}{\sqrt{-2\pi NS''_{N,y_N}(\xi_0^N)}}\big(1 + O(N^{1-3\varepsilon})\big) \tag{A.3}$$

Denote the saddle point of $S_{\lambda,y}$ by ξ_0. Then $\xi_0^N \to \xi_0$ and $S_{N,y_N}(\xi_0^N) \to S_{\lambda,y}(\xi_0) = -R_\lambda(y)$ together with all its derivatives as $N \to \infty$. Notice also that by convexity of ρ and (A.2) the second derivatives $S''_{N,y_N}(\xi_0^N)$ abd $S''_{\lambda,y}(\xi_0)$ are negative.

To estimate I_2 we set $z = e^{i\xi}$ and $R_0^N = e^{i\xi_0^N} \in \mathbb{R}$ then

$$I_2 = \frac{1}{2\pi i}\int\limits_{C_{R_0^N}\setminus C_{\delta_N}} z^{n_N-1}\prod_{i\in\Lambda_N}\big(zm_N^i + (1-m_N^i)\big)\,dz$$

where $C_{R_0^N}$ is the circle of radius R_0^N around zero and C_{δ_N} is its arc $\big\{R_0^N e^{i\xi},\ -\delta_N \le \xi \le \delta_N\big\}$. Let $a \in (0,\frac{1}{2})$ be such that $m(\theta) \in [a, 1-a]$ for all $\theta \in [0,1]$. For any $z \in C_{R_0^N} \setminus C_{\delta_N}$ we obtain

$$\begin{aligned}
&|zm_N^i + (1-m_N^i)|^2 \le \\
&\qquad \le \big(R_0^N m_N^i + (1-m_N^i)\big)^2\left(1 - \frac{2R_0^N m_N^i(1-m_N^i)(1-\cos\delta_N)}{\big(R_0^N m_N^i + (1-m_N^i)\big)^2}\right) \\
&\qquad \le \big(R_0^N m_N^i + (1-m_N^i)\big)^2\big(1 - A\delta_N^2\big)
\end{aligned}$$

where A is a positive constant which does not depend on N. Therefore for our choice of sequence δ_N

$$\begin{aligned}\big|I_2\big| &\leq (R_0^N)^{-n_N}\big(1 - A\delta_N^2\big)^{\frac{N}{2}} \prod_{i\in\Lambda_N}\big(R_0^N m_N^i + (1 - m_N^i)\big) \\ &= e^{NS_{N,y_N}(\xi_0^N)} O\big(e^{-\frac{1}{2}A\delta^2 N^{1-2\varepsilon}}\big) \end{aligned} \tag{A.4}$$

as $N \to \infty$.

From (A.3) and (A.4) we find that

$$P_N(n_N) = \frac{e^{NS_{N,y_N}(\xi_0^N)}}{\sqrt{-2\pi N S''_{N,y_N}(\xi_0^N)}}\big(1 + O(N^{1-3\varepsilon})\big)$$

which immediately implies the statement of the theorem. □

A.2 Continuous case

In this section we formulate and prove theorems which generalize the results presented in Section 3 of [7].

Let φ satisfy (H_1), (H_2'), (H_3) and functions M, ρ, h be defined by (2.1), (2.14), (2.15) respectively. At first, we state two results from [7].

Lemma A.1 ([7]). *Let $f(x)$ be a probability density on $\mathbb{R}$ satisfying the following conditions: $\int xf(x)dx = 0$, $\int x^2 f(x)dx = \sigma^2$, $\int x^4 f(x)dx \leq K$ and $\int |f'(x)|dx \leq K$. Then there is a number $\delta = \delta(K)$ depending only on K such that*

$$\left|\int e^{i\xi x} f(x)dx\right| \leq (1 + |\xi|^2)^{-\delta} \quad \textit{for all} \quad \xi \in \mathbb{R}.$$

Theorem A.2 ([7]). *Let* φ_N *be the density of* $\frac{1}{N}\sum_{i=1}^{N} X_i$*, where* $X_1, \dots, X_N$ *are independent identically distributed random variables with a common density* $e^{-\varphi(x)}$*. Then*

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \log \varphi_N(x) &= -h(x), \\ \lim_{N\to\infty} \frac{1}{N} \frac{\varphi_N'(x)}{\varphi_N(x)} &= -h'(x) \end{aligned}$$

uniformly on compact x*-intervals.*

Let F be a locally Lipschitz continuous function on $\mathbb{R}$ and for some $K > 0$ satisfy

$$C_\alpha = \int_{\mathbb{R}} e^{\alpha F(x) - \varphi(x)} dx < \infty \quad \text{for all } \alpha \in [0, K). \tag{A.5}$$

Define $\varphi_\alpha(x) = -\alpha F(x) + \varphi(x) + \log C_\alpha$. Then $e^{-\varphi_\alpha(x)}$ is a probability density on $\mathbb{R}$ and the integral

$$M_\alpha(\lambda) = \int_{\mathbb{R}} e^{\lambda x - \varphi_\alpha(x)} dx$$

converges for all $\lambda \in \mathbb{R}$ and $\alpha \in [0, K)$. Indeed, by Hölder inequality for any $\varepsilon > 0$ which satisfies the inequality $(1+\varepsilon)\alpha < K$ we have

$$\begin{aligned} C_\alpha \int_{\mathbb{R}} e^{\lambda x - \varphi_\alpha(x)} dx = \int_{\mathbb{R}} e^{\lambda x - \frac{\varepsilon}{1+\varepsilon}\varphi(x)} e^{\alpha F(x) - \frac{1}{1+\varepsilon}\varphi(x)} dx \le \\ \left(\int_{\mathbb{R}} e^{(1+\varepsilon)\alpha F(x) - \varphi(x)} dx \right)^{\frac{1}{1+\varepsilon}} \left(\int_{\mathbb{R}} e^{\frac{1+\varepsilon}{\varepsilon}\lambda x - \varphi(x)} dx \right)^{\frac{\varepsilon}{1+\varepsilon}} < \infty. \end{aligned}$$

If $\alpha = 0$ we always omit the subscript and write simply φ, M etc. which agrees with our previous notation.

Let $d\sigma_N(x)$ be Lebesgue measure on the hyperplane $\pi_N(y) = \{x \in \mathbb{R}^N :$

$\sum_{i=1}^{N} x_i = Ny\}$ and $d\mu_{N,y}(x)$ be given by the density

$$\frac{d\mu_{N,y}(x)}{d\sigma_N(x)} = \frac{\exp\left(-\sum_{i=1}^{N}\varphi(x_i)\right)}{\int\limits_{\pi_N(y)} \exp\left(-\sum_{i=1}^{N}\varphi(x_i)\right) d\sigma_N(x)}.$$

Lemma A.2. *Let $F \in Lip_{loc}(\mathbb{R})$ and satisfies (A.5) for some $K > 0$ and*

$$\int_{\mathbb{R}} |F'(x)| e^{\alpha F(x)-\varphi(x)} dx < \infty \quad \textit{for all} \quad \alpha \in [0, K). \tag{A.6}$$

Then

$$\lim_{N\to\infty} \frac{1}{N} \log \int\limits_{\pi_N(y)} e^{\alpha(F(x_1)+\ldots+F(x_N))} d\mu_{N,y} = \log C_\alpha + h(y) - h_\alpha(y)$$

where $h_\alpha(y) = \sup_z (zy - \log M_\alpha(z))$. The convergence is locally uniform in y and $\alpha \in [0, K)$.

Proof. This statement is an easy consequence of Theorem A.2. We have

$$\frac{1}{N} \log \int\limits_{\pi_N(y)} e^{\alpha(F(x_1)+\ldots+F(x_N))} d\mu_{N,y} =$$

$$\log C_\alpha + \frac{1}{N} \log \int\limits_{\pi_N(y)} e^{-\sum_{i=1}^{N}\varphi_\alpha(x_i)} d\sigma_N - \frac{1}{N} \log \int\limits_{\pi_N(y)} e^{-\sum_{i=1}^{N}\varphi(x_i)} d\sigma_N.$$

The straightforward application of Theorem A.2 to the third term gives $h(y)$. The conditions (A.5) and (A.6) allow us to use the same theorem with φ replaced by φ_α for the second term as well. □

Let λ be a continuous function on $[0, 1]$. Recall that $\lambda_N^i = \lambda\left(\frac{i}{N}\right)$. We denote by $\|\lambda\|$ the sup-norm of λ: $\|\lambda\| = \max_{\theta\in[0,1]} |\lambda(\theta)|$, and by B_R^λ the set $\{\lambda \in C([0, 1]) : \|\lambda\| \leq R\}$.

Define for any $s \in \mathbb{R}$

$$a(x,s) = \frac{1}{M(s)} \exp\left(s(x+\rho'(s)) - \varphi(x+\rho'(s))\right). \tag{A.7}$$

Then

$$\int a(x,s)dx = 1, \quad \int xa(x,s)dx = 0 \tag{A.8}$$

$$\int x^2 a(x,s)dx = \frac{M''(s)}{M(s)} - \left[\frac{M'(s)}{M(s)}\right]^2 = \rho''(s) \tag{A.9}$$

and

$$\int \left|\frac{da(x,s)}{dx}\right| dx \le K = K(s) \tag{A.10}$$

Notice also that

$$\int e^{px} a(x,s)dx = e^{-p\rho'(s)} \frac{M(p+s)}{M(s)} \le C(s,p) \tag{A.11}$$

Theorem A.3 (Local limit theorem). *Let* $X_N^1, \ldots, X_N^N$ *be independent random variables, each* X_N^i *having the density* $a(x, \lambda_N^i)$ *with respect to Lebesgue measure. Let* $a_N^{\lambda(\cdot)}$ *be the density of* $\frac{1}{\sqrt{N}} \sum_{i=1}^N X_N^i$*. Then for any non-negative integer* k *the functions* $a_N^{\lambda(\cdot)}$ *belong to* $C^k(\mathbb{R})$ *for all* $N \ge N_0(\|\lambda\|, k)$ *and*

$$\lim_{N\to\infty} \frac{d^k}{dx^k} a_N^{\lambda(\cdot)}(x) = \frac{1}{\sqrt{2\pi}\sigma(\lambda)} \frac{d^k}{dx^k} e^{-\frac{x^2}{2\sigma^2(\lambda)}} \tag{A.12}$$

uniformly on $\mathbb{R} \times B_R^\lambda$*,* $R < \infty$*. Here* $\sigma^2(\lambda) = \int_0^1 \rho''(\lambda(\theta))d\theta$*.*

Proof. Denote by $\psi_N^i(\xi)$ the characteristic function of X_N^i. Then

$$a_N^{\lambda(\cdot)}(x) = \frac{1}{2\pi} \int e^{-ix\xi} \prod_{i=1}^N \psi_N^i\left(\frac{\xi}{\sqrt{N}}\right) d\xi.$$

We show that

(i) $\prod_{i=1}^{N} \psi_N^i(\frac{\xi}{\sqrt{N}}) \to e^{-\frac{1}{2}\xi^2\sigma^2(\lambda)}$ as $N \to \infty$;

(ii) for any $k \geq 0$ and large enough l the $\sup_{N\geq l} \left|\xi^k \prod_{i=1}^{N} \psi_N^i\left(\frac{\xi}{\sqrt{N}}\right)\right|$ is bounded by an integrable function of ξ uniformly on any $B_R^\lambda,\ R < \infty$.

Then by the dominated convergence theorem and properties of Fouriér transform we conclude that

$$\lim_{N\to\infty} \frac{d^k}{dx^k} a_N^{\lambda(\cdot)}(x) = \frac{d^k}{dx^k}\frac{1}{2\pi}\int e^{-ix\xi}e^{-\xi^2\sigma^2(\lambda)/2}d\xi = \frac{1}{\sqrt{2\pi}\sigma(\lambda)}\frac{d^k}{dx^k}e^{-\frac{x^2}{2\sigma^2(\lambda)}}$$

as claimed.

To prove the convergence (i) we expand $e^{i\xi x}$ in Taylor series and use (A.8) and (A.9). We find

$$\begin{aligned}
\psi_N^i(\frac{\xi}{\sqrt{N}}) &= 1 + \frac{\xi^2}{2N}\int x^2 a(x,\lambda_N^i)dx + \frac{\xi^3}{3!N^{3/2}}\int \theta_1|x|^3 a(x,\lambda_N^i)dx \\
&= 1 - \frac{\xi^2}{2N}\rho''(\lambda_N^i) + \frac{\xi^3}{6N^{3/2}}\int \theta_1|x|^3 a(x,\lambda_N^i)dx,
\end{aligned}$$

where $\theta_1 = \theta_1(y),\ |\theta_1| \leq 1$. The third moment of X_N^i is bounded uniformly in N. Thus

$$\sum_{i=1}^{N} \log\psi_N^i(\frac{\xi}{\sqrt{N}}) = \sum_{i=1}^{N}\log\left(1 - \frac{\xi^2}{2N}\rho''(\lambda_N^i) + O\left(\frac{|\xi|^3}{N^{3/2}}\right)\right)$$

Since $\log(1+z) = z + \theta|z|^2$ with $|\theta| \leq 1$ for $|z| \leq 1/2$ we obtain for all large N that

$$\sum_{i=1}^{N}\log\psi_N^i(\frac{\xi}{\sqrt{N}}) = -\frac{\xi^2}{2N}\sum_{i=1}^{N}\rho''(\lambda_N^i) + O\left(\frac{|\xi|^3}{\sqrt{N}}\right) \ \to\ -\frac{\xi^2}{2}\sigma^2(\lambda)$$

as $N \to \infty$.

For the second part we notice that the properties (A.8), (A.9), (A.10), (A.11) allow us to apply Lemma A.1. Hence there exists $\delta > 0$ which depends only on $\|\lambda\|$ such that

$$\sup_{N\geq l}\left|\xi^k\prod_{i=1}^{N}\psi_N^i\Big(\frac{\xi}{\sqrt{N}}\Big)\right| \leq \sup_{N\geq l}|\xi|^k\left(1+\frac{|\xi|^2}{N}\right)^{-\delta N} \leq |\xi|^k\left(1+\frac{|\xi|^2}{l}\right)^{-\delta l}$$

For large enough l the last function is integrable. This concludes the proof.

□

Theorem A.4 (Large deviations). *Let $Y_N^1,\dots,Y_N^N$ be independent random variables, each Y_N^i having the density*

$$b(y,\lambda_N^i) = \frac{1}{M(\lambda_N^i)}\exp\big(\lambda_N^i y - \varphi(y)\big) \tag{A.13}$$

with respect to Lebesgue measure. Let $b_N^{\lambda(\cdot)}$ be the density of $\frac{1}{N}\sum_{i=1}^{N}Y_N^i$. Define

$$I^{\lambda(\cdot)}(y) = \sup_z\left\{zy - \int_0^1 \log\frac{M(\lambda(\theta)+z)}{M(\lambda(\theta))}d\theta\right\}. \tag{A.14}$$

Then

$$\lim_{N\to\infty}\frac{1}{N}\log b_N^{\lambda(\cdot)}(y) = -I^{\lambda(\cdot)}(y)$$

uniformly on any set of the form $\{|y|\leq R_1\}\times B_{R_2}^{\lambda}$, $R_1,R_2 < \infty$.

Proof.

Step 1. From (A.7) and (A.13) we see that $Y_N^i = X_N^i + \rho'(\lambda_N^i)$ and

$$\frac{1}{N}\sum_{i=1}^{N}Y_N^i = \frac{1}{\sqrt{N}}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}X_N^i\right) + \frac{1}{N}\sum_{i=1}^{N}\rho'(\lambda_N^i).$$

Thus $b_N^{\lambda(\cdot)}(y) = \sqrt{N}a_N^{\lambda(\cdot)}(\sqrt{N}[y - \frac{1}{N}\sum_{i=1}^{N}\rho'(\lambda_N^i)])$. By Theorem A.3 we have

$$\lim_{N\to\infty}\frac{1}{N}\log b_N^{\lambda(\cdot)}\left(\frac{1}{N}\sum_{i=1}^{N}\rho'(\lambda_N^i)\right) = \lim_{N\to\infty}\frac{1}{N}\left(\log\sqrt{N} + \log a_N^{\lambda(\cdot)}(0)\right) = 0.$$

Step 2. Let $z_0 = z_0(y)$ be the point where the supremum in (A.14) is attained and $\tilde{\lambda}(\theta) = \lambda(\theta) + z_0$. Introduce a new set of random variables $\tilde{Y}_N^i$, $i = 1, \ldots, N$, with densities

$$\tilde{b}(y, \tilde{\lambda}_N^i) = \frac{1}{M(\tilde{\lambda}_N^i)} \exp\left(\tilde{\lambda}_N^i y - \varphi(y)\right).$$

Then by Step 1 the density $b_N^{\tilde{\lambda}(\cdot)}$ of the average $\frac{1}{N}\sum_{i=1}^{N} \tilde{Y}_N^i$ satisfies

$$\lim_{N\to\infty} \frac{1}{N} \log b_N^{\tilde{\lambda}(\cdot)} \left(\frac{1}{N} \sum_{i=1}^{N} \rho'(\tilde{\lambda}_N^i) \right) = 0. \tag{A.15}$$

Since $\tilde{b}(y, \tilde{\lambda}_N^i) = b(y, \lambda_N^i) e^{-z_0 y} M(\tilde{\lambda}_N^i)/M(\lambda_N^i)$ using the properties of convolution we can compute that $b_N^{\tilde{\lambda}(\cdot)}(y) = b_N^{\lambda(\cdot)}(y) e^{-z_0 y N} \prod_{i\in\Lambda_N} \left(M(\tilde{\lambda}_N^i)/M(\lambda_N^i)\right)$. From the last equation we obtain

$$\begin{aligned} \frac{1}{N} \log b_N^{\lambda(\cdot)} \left(\frac{1}{N} \sum_{i=1}^{N} \rho'(\tilde{\lambda}_N^i) \right) &= \frac{1}{N} \log b_N^{\tilde{\lambda}(\cdot)} \left(\frac{1}{N} \sum_{i=1}^{N} \rho'(\tilde{\lambda}_N^i) \right) \\ &\quad - z_0 \frac{1}{N} \sum_{i=1}^{N} \rho'(\tilde{\lambda}_N^i) + \frac{1}{N} \sum_{i=1}^{N} \log \frac{M(\tilde{\lambda}_N^i)}{M(\lambda_N^i)}. \end{aligned} \tag{A.16}$$

By our choice of $z_0(y)$ we find

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^{N} \rho'(\tilde{\lambda}_N^i) &= \frac{1}{N} \sum_{i=1}^{N} \frac{M'(\lambda_N^i + z_0)}{M(\lambda_N^i + z_0)} \to \\ &\quad \int_0^1 \frac{M'(\lambda(\theta) + z_0)}{M(\lambda(\theta) + z_0)} d\theta = y \text{ as } N \to \infty. \end{aligned} \tag{A.17}$$

The statement of the theorem follows now from (A.15), (A.16), (A.17) and Lemma A.3 below.

Lemma A.3. *Let*

$$I_N^{\lambda(\cdot)}(y) = -\frac{1}{N} \log b_N^{\lambda(\cdot)}(y)$$

where $b_N^{\lambda(\cdot)}$ *defined as in Theorem A.4. Then for any* $y_1, y_2 \in \{|y| \le R\}$

$$|I_N^{\lambda(\cdot)}(y_1) - I_N^{\lambda(\cdot)}(y_2)| \le C(R, \|\lambda\|)|y_1 - y_2|.$$

Proof. We have

$$I_N^{\lambda(\cdot)}(y) = \frac{1}{N} \log \int\limits_{\pi_N(y)} \prod_{i=1}^{N} \frac{\exp\left(\lambda_N^i x_i - \varphi(x_i)\right)}{M(\lambda_N^i)} d\sigma_N(x) \tag{A.18}$$

Computing the derivative we find

$$\frac{d}{dy} I_N^{\lambda(\cdot)}(y) = -\frac{1}{N} \sum_{i=1}^{N} \lambda_N^i - \frac{1}{N} \int\limits_{\pi_N(y)} \sum_{i=1}^{N} \varphi'(x_i) d\mu_{N,y}^{\lambda(\cdot)}(x). \tag{A.19}$$

where the measure $d\mu_{N,y}^{\lambda(\cdot)}(x)$ is given by

$$\frac{d\mu_{N,y}^{\lambda(\cdot)}(x)}{d\sigma_N(x)} = \frac{\exp\left(\sum\limits_{i=1}^{N}(\lambda_N^i x_i - \varphi(x_i))\right)}{\int\limits_{\pi_N(y)} \exp\left(\sum\limits_{i=1}^{N}(\lambda_N^i x_i - \varphi(x_i))\right) d\sigma_N(x)}.$$

The first term of (A.19) is bounded by $\|\lambda\|$. To estimate the second term we apply Jensen's inequality: for any $\delta > 0$

$$\int\limits_{\pi_N(y)} \sum_{i=1}^{N} |\varphi'(x_i)| d\mu_{N,y}^{\lambda(\cdot)}(x) \le \frac{1}{\delta N} \log \int\limits_{\pi_N(y)} e^{\delta \sum\limits_{i=1}^{N} |\varphi'(x_i)|} d\mu_{N,y}^{\lambda(\cdot)}(x)$$

$$\le \frac{1}{\delta N} \log \frac{\int\limits_{\pi_N(y)} \exp\Big(\sum\limits_{i=1}^{N} \left(\delta|\varphi'(x_i)| + \|\lambda\||x_i| - \varphi(x_i)\right) \Big) d\sigma_N(x)}{\int\limits_{\pi_N(y)} \exp\Big(- \sum\limits_{i=1}^{N} \left(\|\lambda\||x_i| + \varphi(x_i)\right) \Big) d\sigma_N(x)}$$

Let

$$\begin{aligned} \varphi_{\delta,\lambda}(x) &= -\delta|\varphi'(x)| - \|\lambda\||x| + \varphi(x) + \log C_{\delta,\lambda} \\ \varphi_\lambda(x) &= \|\lambda\||x| + \varphi(x) + \log C_\lambda, \end{aligned}$$

where the constants $C_{\delta,\lambda}$ and C_λ are chosen to turn $\exp(-\varphi_{\delta,\lambda})$ and $\exp(-\varphi_\lambda)$ into probability densities. Now we apply Theorem A.2 twice with φ replaced by $\varphi_{\delta,\lambda}$ and φ_λ to see that

$$\lim_{N\to\infty} \frac{1}{\delta N} \log \int\limits_{\pi_N(y)} e^{\delta \sum_{i=1}^{N} |\varphi'(x_i)|} d\mu_{N,y}^{\lambda(\cdot)}(x) \quad = \quad \frac{1}{\delta} \log \frac{C_{\delta,\lambda}}{C_\lambda} \quad + \quad \frac{1}{\delta} \left(h_\lambda(y) - h_{\delta,\lambda}(y)\right)$$

uniformly on compact y-intervals. Rate functions $h_{\delta,\lambda}$ and h_λ are defined as in (2.15) with a suitable choice of index for φ. They are bounded on bounded y-intervals. This concludes the proof. □

Appendix B

One lemma from convex analysis

Lemma B.1. *Let f, g be non-negative continuous functions and μ be a probability measure on $\mathbb{R}^n$. Assume also that $\int g\,d\mu = 1$. Then for any $p \in [1,2]$*

$$\int f^p g\,d\mu \leq 2\log^p[\,C_p(\int e^f\,d\mu + 1)] + 6\int g\,|\log g|^p\,d\mu$$

where $C_p \leq 1 + 2^{p-1}$.

Proof. Denote $\frac{1}{p}$ by α. Then $\alpha \in [\frac{1}{2}, 1]$. The statement of the lemma is mainly a consequence of convexity of the function

$$F(u) = e^{u^\alpha} - u^\alpha \qquad \text{for } u \geq 0.$$

Step 1. Jensen's inequality implies that

$$F(\int (f - \log g)_+^p g\,d\mu) \leq \int F((f - \log g)_+^p) g\,d\mu$$

where $(y)_+ = \max\{y, 0\}$. Substituting a formula for F and using the fact that $u^p \le e^u$ for $u \ge 0$ and $p \in [1, 2]$ we obtain

$$
\begin{aligned}
1 &\le \exp\left(\int (f - \log g)_+^p g \, d\mu\right)^{\frac{1}{p}} \\
&\le \int\limits_{f \ge \log g} e^f \, d\mu + \int\limits_{f < \log g} g \, d\mu + \left(\int (f - \log g)_+^p g \, d\mu\right)^{\frac{1}{p}} \\
&\le \int\limits_{f \ge \log g} e^f \, d\mu + \int\limits_{f < \log g} g \, d\mu + \Big(\int\limits_{f \ge \log g} e^f \, d\mu + \int\limits_{f < \log g} g \, d\mu \Big)^{\frac{1}{p}} \\
&= \lambda^p + \lambda
\end{aligned}
\tag{B.1}
$$

where $\lambda = \Big(\int\limits_{f \ge \log g} e^f \, d\mu + \int\limits_{f < \log g} g \, d\mu \Big)^{\frac{1}{p}} \ge 0$. Looking at the first and the last terms of (B.1) we see that

$$\lambda^p + \lambda - 1 \ge 0. \tag{B.2}$$

This implies that λ is bounded away from 0 and thus there is a constant $C = C(p)$ such that $\lambda \le C\lambda^p$. It is easy to see that the only positive root of the left hand side of (B.2) is greater or equal than $\frac{1}{2}$. Therefore $C \le 2^{p-1}$ and

$$\lambda^p + \lambda \le (1 + 2^{p-1})\lambda^p = C_p \lambda^p.$$

From the above estimate and (B.1) we find

$$\exp\Big(\int (f - \log g)_+^p g \, d\mu\Big)^{\frac{1}{p}} \le C_p \Big(\int\limits_{f \ge \log g} e^f \, d\mu + \int\limits_{f < \log g} g \, d\mu \Big).$$

After applying the function $\log^p u$ to both sides of the above inequality we arrive at

$$\int (f-\log g)_+^p g\,d\mu \le \log^p \Big[C_p\big(\int\limits_{f\ge \log g} e^f\,d\mu + \int\limits_{f<\log g} g\,d\mu\ \big)\Big]$$

$$\le \log^p \Big[C_p\big(\int\limits_{f\ge \log g} e^f\,d\mu + 1\ \big)\Big]. \tag{B.3}$$

Step 2. We estimate the left hand side of (B.3) from below using the fact that $|a-b|^p \ge ||a|-|b||^p \ge |a|^p + |b|^p - 2|a|^{\frac{p}{2}}|b|^{\frac{p}{2}}$. The latter can be easily proved by squaring the inequality

$$(1-x)^{\frac{p}{2}} \ge 1 - x^{\frac{p}{2}} \qquad \text{for } 0 \le x \le 1$$

and setting $x = \left|\frac{a}{b}\right|$. On the set $\{f \ge \log g\}$ we have

$$(f-\log g)_+^p = |f-\log g|^p \ge f^p + |\log g|^p - 2f^{\frac{p}{2}}|\log g|^{\frac{p}{2}}$$

which implies that

$$\int\limits_{f\ge \log g} f^p g\,d\mu \le \int (f-\log g)_+^p g\,d\mu + 2\int\limits_{f\ge\log g} f^{\frac{p}{2}}|\log g|^{\frac{p}{2}} g\,d\mu$$

and therefore

$$\begin{aligned}\int f^p g\,d\mu &= \int\limits_{f\ge\log g} f^p g\,d\mu + \int\limits_{f<\log g} f^p g\,d\mu\\ &\le \log^p\Big[C_p\big(\int\limits_{f\ge\log g} e^f\,d\mu + 1\big)\Big] + 2\int\limits_{f\ge\log g} f^{\frac{p}{2}}|\log g|^{\frac{p}{2}} g\,d\mu + \int |\log g|^p g\,d\mu\\ &\le \log^p\Big[C_p\big(\int\limits_{f\ge\log g} e^f\,d\mu + 1\big)\Big] + 2\Big(\int |\log g|^p g\,d\mu\Big)^{\frac{1}{2}}\Big(\int f^p g\,d\mu\Big)^{\frac{1}{2}}\\ &\qquad + \int |\log g|^p g\,d\mu.\end{aligned}$$

This is a quadratic inequality in $\beta = \left(\int f^p g\, d\mu\right)^{\frac{1}{2}}$ of the type

$$\beta^2 \leq \gamma^2 + 2\delta\beta + \delta^2, \qquad \beta, \gamma, \delta \geq 0$$

where $\gamma = \log^{\frac{p}{2}} \left[C_p\left(\int\limits_{f \geq \log g} e^f\, d\mu + 1\right)\right]$ and $\delta = \left(\int |\log g|^p g\, d\mu\right)^{\frac{1}{2}}$. It gives us a bound

$$\beta \leq \delta + \sqrt{2\delta^2 + \gamma^2},$$

. By the Hölder inequality $\beta^2 \leq 2\gamma^2 + 6\delta^2$. Substituting back the expressions for $\beta,\ \gamma$ and δ we obtain the statement of the lemma. $\square$

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