

# The behavior of the specific entropy in the hydrodynamic scaling limit for Ginzburg-Landau model

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August 28, 2005

## Abstract

The paper studies the behavior of the specific entropy for Ginzburg-Landau type models under the hydrodynamical scaling of time and space. It is shown that if the initial configurations possess a macroscopic profile then for any fixed positive macroscopic time the specific microscopic entropy converges to the macroscopic entropy. The latter is defined in terms of the solution of the corresponding hydrodynamical equation, which is a non-linear diffusion equation. The above result is equivalent to the following statement: under the hydrodynamical scaling for any positive macroscopic time the specific microscopic entropy relative to local Gibbs measures converges to zero.

## 1 The description of the model

Let  $\Lambda_N$  be the one-dimensional periodic integer lattice  $\mathbb{Z}/N\mathbb{Z}$ ,  $N \in \mathbb{N}$ . The variable  $x_i \in \mathbb{R}$ ,  $i = 1, \dots, N$ , attached to a lattice site, represents a charge at this site. The collection of all charges is a vector  $x^{(N)}$  in  $\mathbb{R}^N$ , which we call the configuration. The nearest neighbor charges interact and the

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AMS 1991 *subject classifications*. Primary 60K35; secondary 82C22.

*Key words and phrases*. Entropy, Ginzburg-Landau model, local Gibbs measures, hydrodynamical scaling limit, hydrodynamical equation.

configuration  $x^{(N)}$  changes in time. Applying the diffusive scaling of space and time, i. e. shrinking the spacing between charges by  $N$  and speeding up the time by  $N^2$ , we obtain a system of charges located at points  $i/N$ ,  $i = 1, \dots, N$ , of the circle  $S = \mathbb{R}/\mathbb{Z}$ . The evolution of  $x^{(N)}(t)$  is described by the following system of stochastic differential equations:

$$dx_i(t) = \frac{N^2}{2} \left( \varphi'(x_{i+1}(t)) - 2\varphi'(x_i(t)) + \varphi'(x_{i-1}(t)) \right) dt + N (d\beta_i(t) - d\beta_{i+1}(t)), \quad i \in \Lambda_N, \quad (1)$$

where  $\beta_i$ ,  $i = 1, \dots, N$ , are independent Brownian motions and  $\varphi \in C^2(\mathbb{R})$  satisfies the hypotheses  $(H_1) - (H_3)$  listed below.

$$(H_1) \int_{\mathbb{R}} e^{-\varphi(x)} dx = 1.$$

$$(H_2) \int_{\mathbb{R}} e^{\alpha|\varphi'(x)| - \varphi(x)} dx < \infty \text{ for any } \alpha > 0.$$

$(H_3)$  There exist positive constants  $C_0, C_1, C_2, R$  and  $p \in [1/2, 1]$  such that

$$\varphi''(x) \geq C_0 \quad \text{for all } |x| \geq R; \quad (2)$$

$$\varphi''(x) \leq C_1 \varphi(x) + C_2 \quad \text{for all } x \in \mathbb{R}; \quad (3)$$

$$|\varphi'(x)| \leq C_1 (|\varphi(x)|^p + 1) \quad \text{for all } x \in \mathbb{R}. \quad (4)$$

For a simple example one can take  $\varphi(x) = (x^2 - \log 2\pi)/2$ .

The convexity assumption (2) implies the convergence of the integral

$$(H'_3) M(\lambda) \stackrel{\text{def}}{=} \int_{\mathbb{R}} e^{\lambda x - \varphi(x)} dx < \infty \text{ for any } \lambda \in \mathbb{R}.$$

The infinitesimal generator of the diffusion process  $x^{(N)}(t)$  is given by

$$\begin{aligned} \mathcal{L}_N = & \frac{N^2}{2} \sum_{i \in \Lambda_N} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right)^2 \\ & - \frac{N^2}{2} \sum_{i \in \Lambda_N} (\varphi'(x_i) - \varphi'(x_{i+1})) \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}} \right). \end{aligned} \quad (5)$$

The generator  $\mathcal{L}_N$  is formally symmetric with respect to the product measure

$$d\mu_N(x) = \exp\left(-\sum_{i \in \Lambda_N} \varphi(x_i)\right) dx_1 \dots dx_N \quad (6)$$

on  $\mathbb{R}^N$  which is an invariant but non-ergodic measure for our diffusion. Non-ergodicity follows from the fact that  $\mathcal{L}_N$  is degenerate in the direction of vector  $(1, 1, \dots, 1)$ .

For  $n \in \mathbb{N}$  and  $y \in \mathbb{R}$  let

$$\mathbb{X}_{n,y} = \left\{x \in \mathbb{R}^k : \sum_{i=1}^n x_i = ny\right\}. \quad (7)$$

Denote by  $\mathcal{L}_{N,y}$  the restriction of  $\mathcal{L}_N$  to the hyperplane  $\mathbb{X}_{N,y}$ . Then the operator  $\mathcal{L}_{N,y}$  is strictly elliptic. The conditional measure

$$\mu_{N,y}(\cdot) = \mu_N(\cdot \mid \sum_{i \in \Lambda_N} x_i^{(N)} = Ny), \quad (8)$$

supported on  $\mathbb{X}_{N,y}$ , is invariant and ergodic.

The Dirichlet form  $D_{N,y}$  corresponding to  $\mathcal{L}_{N,y}$  is defined by the relation

$$D_{N,y}(f) \stackrel{\text{def}}{=} - \int_{\mathbb{X}_{N,y}} f \mathcal{L}_{N,y} f d\mu_{N,y} = \frac{N^2}{2} \int_{\mathbb{X}_{N,y}} \sum_{i \in \Lambda_N} \left( \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_{i+1}} \right)^2 d\mu_{N,y}. \quad (9)$$

It is non-degenerate for all non-constant  $f \in W^{1,2}(\mathbb{X}_{N,y}, \mu_{N,y})$ .

Assume that the process  $x^{(N)}(\cdot)$  starts from a deterministic configuration  $\eta^{(N)} = (\eta_1^{(N)}, \dots, \eta_N^{(N)})$ . Denote by  $P^{(N)}$  the measure on the space of continuous paths corresponding to the process  $x^{(N)}(\cdot)$ . Let  $\nu_N^t$  be the distribution of charges in  $\mathbb{R}^N$  at time  $t > 0$ . Since the total charge  $\sum_{i \in \Lambda_N} x_i^{(N)}$  is preserved in time, the measure  $\nu_N^t$  is supported on the hyperplane  $\mathbb{X}_{N,y_N}$ , where

$$y_N = \frac{1}{N} \sum_{i \in \Lambda_N} \eta_i^{(N)}.$$

Restricting  $\nu_N^t$  to  $\mathbb{X}_{N,y_N}$  we obtain a measure  $\nu_{N,y_N}^t$ , which is absolutely continuous with respect to  $\mu_{N,y_N}$ . Let

$$f_{N,y_N}^t \stackrel{\text{def}}{=} \frac{d\nu_{N,y_N}^t}{d\mu_{N,y_N}}.$$

The density  $f_{N,y_N}^t$  (up to a normalization constant) is the restriction to  $\mathbb{X}_{N,y_N}$  of the solution to the following Cauchy problem:

$$\begin{aligned} \frac{\partial f}{\partial t} &= \mathcal{L}_N f \quad \text{on } (0, +\infty) \times \mathbb{R}^N \\ f|_{t=0} &= \delta_{\eta^{(N)}}. \end{aligned}$$

It is a consequence of ellipticity of  $\mathcal{L}_{N,y_N}$  that  $f_{N,y_N}^t$  is a smooth and strictly positive function on  $\mathbb{X}_{N,y_N}$  for all  $t > 0$ .

For any  $t > 0$  the microscopic entropy  $H(\nu_{N,y_N}^t | \mu_{N,y_N})$  relative to the invariant measure is given by

$$\begin{aligned} H(\nu_{N,y_N}^t | \mu_{N,y_N}) &= \int_{\mathbb{X}_{N,y_N}} \log \left( \frac{d\nu_{N,y_N}^t}{d\mu_{N,y_N}} \right) d\nu_{N,y_N}^t \\ &= \int_{\mathbb{X}_{N,y_N}} f_{N,y_N}^t \log f_{N,y_N}^t d\mu_{N,y_N}. \end{aligned}$$

Clearly, for our initial data  $H(\nu_{N,y_N}^0 | \mu_{N,y_N}) = +\infty$ .

The relative entropy  $H(\nu_{N,y_N}^t | \mu_{N,y_N})$  is decreasing in time: for  $t > 0$

$$\begin{aligned} \frac{dH(\nu_{N,y_N}^t | \mu_{N,y_N})}{dt} &= -\frac{N^2}{2} \int_{\mathbb{X}_{N,y_N}} \frac{1}{f_{N,y_N}^t} \sum_{i \in \Lambda_N} \left( \frac{\partial f_{N,y_N}^t}{\partial x_i} - \frac{\partial f_{N,y_N}^t}{\partial x_{i+1}} \right)^2 d\mu_{N,y_N} \\ &= -4D_{N,y_N} \left( \sqrt{f_{N,y_N}^t} \right) < 0. \end{aligned} \quad (10)$$

It was shown in [4] that if the average  $N^{-1} \sum_{i \in \Lambda_N} \varphi(\eta_i^{(N)})$  is bounded uniformly in  $N$ , and  $\varphi$  satisfies  $(H_1)$ ,  $(H_2)$ ,  $(H_3)$ , and the inequalities

$$-C \leq \varphi''(x) \leq C_1 \varphi(x) + C_2, \quad x \in \mathbb{R}, \quad (11)$$

for some positive constants  $C$ ,  $C_1$ , and  $C_2$ , then

$$H(\nu_{N,y_N}^t | \mu_{N,y_N}) \leq C(t)N \quad \text{for any } t > 0. \quad (12)$$

Moreover, for any sequence  $\{t_N\}$  such that  $t_N \rightarrow 0$  as  $N \rightarrow \infty$  and  $N^2 t_N$  is bounded away from 0 the corresponding constants  $C(t_N)$  in (12) are still uniformly bounded in  $N$ .

We study the behavior of the specific entropy  $N^{-1}H(\nu_{N,y_N}^t | \mu_{N,y_N})$  as  $N \rightarrow \infty$ . We prove that it converges to the macroscopic entropy as  $N \rightarrow \infty$  (Theorem 2.2). We also discuss the equivalent result about the behavior of the specific entropy relative to local Gibbs measures (Theorem 2.3). The necessary definitions and the precise formulations of these results are given in Section 2. We obtain a lower bound on  $N^{-1}H(\nu_{N,y_N}^t | \mu_{N,y_N})$  in Section 3. The proof of the upper bound is given in Section 4 modulo a number of technical lemmas, which we prove in Sections 5 and 6. We show the equivalence of Theorem 2.2 and Theorem 2.3 in Section 7. Appendices A and B contain limit theorems for densities, which we use throughout the paper, and a lemma from convex analysis, which is needed in Section 6.

## 2 The existence of the scaling limit and main results

At first we give a definition of an asymptotic macroscopic profile.

**Definition 2.1.** *For each  $N \in \mathbb{N}$  let  $\nu_N$  be a probability measure on  $\mathbb{R}^N$ . We shall say that the set  $\{\nu_N\}$  possesses an asymptotic macroscopic profile  $m \in L^1(S)$  as  $N \rightarrow \infty$  and denote this by  $\nu_N \sim m$  if for every function  $J \in C(S)$  and each  $\delta > 0$*

$$\lim_{N \rightarrow \infty} \nu_N \left\{ \eta \in \mathbb{R}_N : \left| \frac{1}{N} \sum_{i \in \Lambda_N} J\left(\frac{i}{N}\right) \eta_i - \int_S J(\theta) m(\theta) d\theta \right| > \delta \right\} = 0. \quad (13)$$

Our main assumption on the initial data is the existence of an asymptotic macroscopic profile, i. e.

$$\nu_N^0 = \delta_{\eta^{(N)}} \sim m_0 \quad (14)$$

for some  $m_0 \in L^1(S)$ . In particular,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i \in \Lambda_N} \eta_i^{(N)} = \int_0^1 m_0(\theta) d\theta \stackrel{\text{def}}{=} \bar{m}. \quad (15)$$

Then it is known (see Theorem 2.1 below) that for any  $t > 0$

$$\nu_{N,y_N}^t \sim m(t, \cdot),$$

where  $m(t, \cdot)$  is the solution of a non-linear diffusion equation with the initial condition  $m_0$ . We shall formulate this statement more precisely after introducing some notation.

Let  $M$  be defined as in  $(H'_3)$  above. It is easy to check that

$$\rho = \log M(\lambda) \tag{16}$$

is a convex analytic function. Its convex conjugate

$$h(y) = \sup_{z \in \mathbb{R}} (yz - \log M(z)) \tag{17}$$

is also analytic and

$$h'^{-1}(\lambda) = \frac{M'(\lambda)}{M(\lambda)} = \rho'(\lambda). \tag{18}$$

The following theorem was proven in [4].

**Theorem 2.1.** *Assume that  $\varphi$  satisfies  $(H_1)$ ,  $(H_2)$ ,  $(H'_3)$ , and (11). Let the sequence  $\eta^{(N)}$  of deterministic configurations satisfy (14) for some  $m_0 \in L^1(S)$  and the average  $N^{-1} \sum_{i \in \Lambda_N} \varphi(\eta_i^{(N)})$  be bounded uniformly in  $N$ . Let  $m$  be a weak solution of the equation*

$$\frac{\partial m}{\partial t} = \frac{1}{2} \frac{\partial^2 h'(m)}{\partial \theta^2}, \quad (t, \theta) \in (0, \infty) \times S, \tag{19}$$

with the initial condition  $m_0$ . Then  $\nu_{N, y_N}^t \sim m(t, \cdot)$  locally uniformly in  $t > 0$ .

**Remark.** The above theorem is the extension of the results obtained earlier in [3] to the case of deterministic initial data. The main ingredient of the proof of Theorem 2.1 is the entropy estimate (12).

The main result of this paper is the following theorem.

**Theorem 2.2.** *Let the sequence  $\eta^{(N)}$ ,  $N = 1, 2, \dots$ , of deterministic configurations possess an asymptotic profile  $m_0 \in L^1(S)$  as  $N \rightarrow \infty$  in the sense of (14). Assume that  $\varphi$  satisfies  $(H_1) - (H_3)$  and that there are constants  $C > 0$  and  $\delta_0 > 0$  such that*

$$\sum_{i \in \Lambda_N} \varphi^{2p+\delta_0}(\eta_i^{(N)}) \leq CN, \tag{20}$$

where  $p$  is given in  $(H_3)$ . Denote the average  $N^{-1} \sum_{i \in \Lambda_N} \eta_i^{(N)}$  by  $y_N$ . Then for any  $t > 0$  the specific microscopic entropy converges to the macroscopic entropy as  $N \rightarrow \infty$ , namely

$$\lim_{N \rightarrow \infty} \frac{1}{N} H(\nu_{N,y_N}^t | \mu_{N,y_N}) = \int_S h(m(t, \theta)) d\theta - h \left( \int_S m(t, \theta) d\theta \right).$$

where  $m(t, \theta)$  is the solution of (19) with the initial condition  $m_0$ .

**Remark.** Notice that  $\int_S m(t, \theta) d\theta$  does not depend on  $t$  and is equal to  $\bar{m}$  (see (15)).

To every solution  $m$  of the hydrodynamical equation (19) we can associate a family of local Gibbs measures  $\gamma_N^{m_t}$  on  $\mathbb{R}^N$ . Let  $\lambda(t, \theta) = h'(m(t, \theta))$ . Measures  $\gamma_N^{m_t}$  are defined by their densities  $g_N^{m_t}$  with respect to the invariant measure  $\mu_N$ :

$$g_N^{m_t}(x) = \frac{d\gamma_N^{m_t}}{d\mu_N} = \frac{1}{Z_N^{m_t}} \exp \left( \sum_{i \in \Lambda_N} \lambda \left( t, \frac{i}{N} \right) x_i \right)$$

where  $Z_N^{m_t}$  is the normalization constant:  $Z_N^{m_t} = \prod_{i \in \Lambda_N} M \left( \lambda \left( t, \frac{i}{N} \right) \right)$ . Measures  $\gamma_N^{m_t}$  depend only on the solution  $m(t, \theta)$  and form a convenient set of reference measures. The conditional measure  $\gamma_N^{m_t}(\cdot | \sum_{i \in \Lambda_N} x_i^{(N)} = Ny)$  and its density relative to  $\mu_{N,y}$  will be denoted by  $\gamma_{N,y}^{m_t}(\cdot)$  and  $g_{N,y}^{m_t}$  respectively.

It is not difficult to show that for any  $t \geq 0$

$$\gamma_N^{m_t} \sim m(t, \cdot).$$

If we start the process from the local equilibrium  $\gamma_N^{m_0}$  then the measure  $\psi_N^t$ , which describes the distribution of this process at time  $t$ , will diverge from  $\gamma_N^{m_t}$ . But the specific entropy

$$\frac{1}{N} H(\psi_N^t | \gamma_N^{m_t})$$

approaches zero as  $N \rightarrow \infty$ . This result was proven in [7] as a part of the derivation of the scaling limit for this case.

The next theorem shows that even if the process starts very far away from the local equilibrium, the specific entropy relative to local Gibbs measures converges to zero as  $N \rightarrow \infty$  for any  $t > 0$ . Thus, as far as the specific entropy is concerned, the process with the deterministic initial data in the hydrodynamical scaling limit immediately becomes indistinguishable from the one, which started from the local equilibrium.

**Theorem 2.3.** *Under the assumptions of Theorem 2.2 for any fixed positive macroscopic time  $t$  the specific relative entropy*

$$\frac{1}{N} H(\nu_{N,y_N}^t | \gamma_{N,y_N}^{m_t}) \stackrel{\text{def}}{=} \frac{1}{N} \int_{\mathbb{X}_{N,y_N}} f_{N,y_N}^t \log \frac{f_{N,y_N}^t}{g_{N,y_N}^t} d\mu_{N,y_N}$$

*approaches zero as  $N \rightarrow \infty$ .*

We show (see Lemma 7.1) that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \int_{\mathbb{X}_{N,y_N}} f_{N,y_N}^t \log g_{N,y_N}^t d\mu_{N,y_N} = \int_0^1 h(m(t, \theta)) d\theta - h(\bar{m}).$$

This implies the equivalence of Theorem 2.2 and Theorem 2.3.

We prove Theorem 2.2 by establishing the lower and upper bounds on the specific macroscopic entropy as  $N \rightarrow \infty$ .

### 3 Lower bound.

We show that

$$\liminf_{N \rightarrow \infty} \frac{1}{N} H(\nu_{N,y_N}^t | \mu_{N,y_N}) \geq \int_0^1 h(m(t, \theta)) d\theta - h(\bar{m}). \quad (21)$$

**Lemma 3.1.** *Let  $\varphi$  satisfy  $(H_1)$ ,  $(H_3')$ . Assume that a sequence of real numbers  $\{y_N\}$  and probability measures  $\{\nu_N\}$  satisfy the following conditions:*

- (a) *for each  $N$  measure  $\nu_N$  is supported on  $\mathbb{X}_{N,y_N}$  (see (7));*
- (b)  *$H(\nu_N | \mu_{N,y_N}) \leq CN$ , where  $\mu_{N,y_N}$  is defined by (8);*
- (c)  *$\nu_N \sim m$  for some function  $m$  such that  $h \circ m \in L^1(S)$ .*

*Then*

$$\liminf_{N \rightarrow \infty} \frac{1}{N} H(\nu_N | \mu_{N,y_N}) \geq \int_S h(m(\theta)) d\theta - h\left(\int_S m(\theta) d\theta\right).$$

**Proof.** The entropy inequality implies that for every  $J \in C(S)$

$$\frac{1}{N} H(\nu_N | \mu_{N, y_N}) \geq \frac{1}{N} E_{\nu_N} \sum_{i \in \Lambda_N} J\left(\frac{i}{N}\right) x_i - \frac{1}{N} \log E_{\mu_{N, y_N}} \exp\left(\sum_{i \in \Lambda_N} J\left(\frac{i}{N}\right) x_i\right). \quad (22)$$

Condition (c) means that

$$\frac{1}{N} \sum_{i \in \Lambda_N} J\left(\frac{i}{N}\right) x_i \rightarrow \int_S J(\theta) m(\theta) d\theta$$

as  $N \rightarrow \infty$  in probability relative to  $\nu_N$ . Next we prove the uniform integrability of  $N^{-1} \sum_{i \in \Lambda_N} |x_i|$  under  $\nu_N$ , which allows us to conclude that

$$\lim_{N \rightarrow \infty} \frac{1}{N} E_{\nu_N} \sum_{i \in \Lambda_N} J\left(\frac{i}{N}\right) x_i = \int_S J(\theta) m(\theta) d\theta.$$

**Lemma 3.2.** *Let  $\varphi$  satisfy  $(H_1)$  and  $(H_3')$ . Define*

$$A_{N,R} = \left\{ x \in \mathbb{X}_{N, y_N} : N^{-1} \sum_{i \in \Lambda_N} |x_i| > R \right\}.$$

*Then under assumptions (a) and (b) of Lemma 3.1*

$$\lim_{R \rightarrow \infty} \int_{A_{N,R}} \frac{1}{N} \sum_{i \in \Lambda_N} |x_i| d\nu_N = 0$$

*uniformly in  $N$ .*

**Proof.** It can be shown (see, for example, [6]) that  $(H_3')$  implies the existence of a function  $\omega : [0, \infty) \rightarrow [1, \infty)$ , which is symmetric, convex and satisfies

$$\lim_{|x| \rightarrow \infty} \frac{|x|}{\omega(|x|)} = 0 \quad \text{and} \quad \int_{\mathbb{R}} e^{\omega(|x|) - \varphi(x)} dx < \infty. \quad (23)$$

For any  $\varepsilon > 0$  and all sufficiently large  $R$  we obtain

$$\begin{aligned} \int_{A_{N,R}} N^{-1} \sum_{i \in \Lambda_N} |x_i| d\nu_N &\leq \\ &\leq \varepsilon \int_{A_{N,R}} \omega\left(N^{-1} \sum_{i \in \Lambda_N} |x_i|\right) d\nu_N \leq \varepsilon \int_{A_{N,R}} N^{-1} \sum_{i \in \Lambda_N} \omega(|x_i|) d\nu_N \leq \\ &\leq \varepsilon N^{-1} \log \int_{\mathbb{X}_{N,y_N}} \exp\left(\sum_{i \in \Lambda_N} \omega(|x_i|)\right) d\mu_{N,y_N} + \varepsilon N^{-1} H(\nu_N | \mu_{N,y_N}) \leq C\varepsilon \end{aligned}$$

uniformly in  $N$ . This follows from (23), Theorem A.1 and condition (b) of Lemma 3.1. Hence

$$\lim_{R \rightarrow \infty} \int_{A_{N,R}} N^{-1} \sum_{i \in \Lambda_N} |x_i| d\nu_N = 0$$

uniformly in  $N$ . □

To compute the limit of the second term in the right-hand side of (22) we notice at first that by assumption (c) we have

$$\lim_{N \rightarrow \infty} y_N = \int_S m(\theta) d\theta.$$

It follows from Theorems A.1 and A.2 that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \log E_{\mu_{N,y_N}} \exp\left(\sum_{i \in \Lambda_N} J\left(\frac{i}{N}\right) x_i\right) &\leq \\ &\leq \int_S \log M(J(\theta)) d\theta + h\left(\int_S m(\theta) d\theta\right), \end{aligned}$$

and, therefore,

$$\begin{aligned} \liminf_{N \rightarrow \infty} \frac{1}{N} H(\nu_N | \mu_{N,y_N}) &\geq \\ &\geq \sup_{J \in C(S)} \left( \int_S J(\theta) m(\theta) - \log M(J(\theta)) d\theta \right) - h\left(\int_S m(\theta) d\theta\right) = \\ &= \int_S h(m(\theta)) d\theta - h\left(\int_S m(\theta) d\theta\right), \end{aligned}$$

as claimed. □

The lower bound (21) follows immediately from (12), Theorem 2.1 and Lemma 3.1.

## 4 Upper bound.

We have to show that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} H(\nu_{N, y_N}^t | \mu_{N, y_N}) \leq \int_S h(m(t, \theta)) d\theta - h(\bar{m}) \quad (24)$$

The proof depends on a number of lemmas. The precise statements of these lemmas are given in appropriate steps below but the proofs are presented in the next two sections.

**Step 1.** We start with a martingale decomposition.

**Lemma 4.1 (Martingale Decomposition).** *Let  $(\Omega, \mathcal{G}_0, P)$  be a probability space and  $\mathcal{G}_0 \supset \mathcal{G}_1 \supset \dots \supset \mathcal{G}_k$  be a decreasing sequence of  $\sigma$ -algebras. For a non-negative random variable  $X_0$ , such that  $E X_0 |\log X_0| < \infty$ , define  $X_i = E(X_0 | \mathcal{G}_i)$ ,  $i = 0, 1, \dots, k$ . Then*

$$E X_0 \log X_0 = \sum_{i=1}^k E E \left( X_{i-1} \log \frac{X_{i-1}}{X_i} | \mathcal{G}_i \right) + E X_k \log X_k.$$

The proof is straightforward, and we omit it.

We apply the above lemma in the following context. Consider  $f_{N, y_N}^t$  as a random variable on the probability space  $(\mathbb{X}_{N, y_N}, \mathcal{G}_0, \mu_{N, y_N})$ , where  $\mathcal{G}_0$  is the Borel  $\sigma$ -algebra on  $\mathbb{X}_{N, y_N}$ . Fix a large number  $l \in \mathbb{N}$ . Divide the circle  $S$  into  $k_N = \lfloor N/(2l+1) \rfloor$  equal parts, which we call “boxes”. Each box contains  $(2l+1)$  or  $(2l+2)$  sites. Let  $B_i \subset \Lambda_N$  be the set of indexes corresponding to the  $i$ -th box and

$$\bar{x}_i = \frac{1}{|B_i|} \sum_{j \in B_i} x_j, \quad i = 1, \dots, k_N$$

where  $|B_i|$  is the number of elements in  $B_i$ . For  $i = 1, \dots, k_N$  define

$$\mathcal{G}_i = \left\{ \sigma\text{-subalgebra of } \mathcal{G}_0 \text{ generated by the averages } \bar{x}_1, \bar{x}_2, \dots, \bar{x}_i \text{ and} \right. \\ \left. \text{by } x_j, j \in \bigcup_{n=i+1}^{k_N} B_n \right\}.$$

Let  $f_i^t = E_{\mu_{N,y_N}}(f_{N,y_N}^t | \mathcal{G}_i)$ . The dependence of  $\mathcal{G}_i$  and  $f_i^t$  on  $N$  will not be reflected in the notation. By Lemma 4.1

$$\begin{aligned} \frac{1}{N} H(\nu_{N,y_N}^t | \mu_{N,y_N}) &= \frac{1}{N} E_{\mu_{N,y_N}} f_{N,y_N}^t \log f_{N,y_N}^t \\ &= \frac{1}{N} \sum_{i=1}^{k_N} E_{\mu_{N,y_N}} E_{\mu_{N,y_N}} \left( f_{i-1}^t \log \frac{f_{i-1}^t}{f_i^t} | \mathcal{G}_i \right) + \frac{1}{N} E_{\mu_{N,y_N}} f_{k_N}^t \log f_{k_N}^t. \end{aligned}$$

Since the entropy decreases in time we can average in  $t$  and obtain that for any  $t_0 \in (0, t)$

$$\begin{aligned} \frac{1}{N} H(\nu_{N,y_N}^t | \mu_{N,y_N}) &\leq \frac{1}{N(t-t_0)} \int_{t_0}^t H(\nu_{N,y_N}^s | \mu_{N,y_N}) ds \\ &= \frac{1}{N(t-t_0)} \int_{t_0}^t \sum_{i=1}^{k_N} E_{\mu_{N,y_N}} E_{\mu_{N,y_N}} \left( f_{i-1}^s \log \frac{f_{i-1}^s}{f_i^s} | \mathcal{G}_i \right) ds \\ &\quad + \frac{1}{N(t-t_0)} \int_{t_0}^t E_{\mu_{N,y_N}} f_{k_N}^s \log f_{k_N}^s ds. \quad (25) \end{aligned}$$

**Step 2.** We show that the first term in the right-hand side of (25) can be made arbitrarily small as  $N \rightarrow \infty$ .

At first we apply a version of the logarithmic Sobolev inequality (see Theorem 4.1 below) to each term of the sum in the right-hand side of (25).

Let  $n \in \mathbb{N}$ ,  $y \in \mathbb{R}$  and  $\mathbb{X}_{n,y}$  be defined by (7). Denote the Lebesgue measure on  $\mathbb{X}_{n,y}$  by  $\sigma_{n,y}$ .

**Theorem 4.1.** *Let  $\varphi \in C^2(\mathbb{R})$  satisfies  $(H_1)$  and the convexity condition (2). Define the probability measure  $\mu_{n,y}$  on the hyperplane  $\mathbb{X}_{n,y}$  by*

$$d\mu_{n,y} = \frac{1}{Z_{n,y}} e^{-\sum_{i=1}^l \varphi(x_i)} d\sigma_{n,y},$$

where  $Z_{n,y}$  is the normalization constant. Then for any smooth  $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that  $E_{\mu_{n,y}} f = 1$  we have the inequality

$$E_{\mu_{n,y}} f \log f \leq C(n, C_0, R) \int \sum_{i=1}^{n-1} \left( \frac{\partial \sqrt{f}}{\partial x_i} - \frac{\partial \sqrt{f}}{\partial x_{i+1}} \right)^2 d\mu_{n,y}, \quad (26)$$

where  $R$  is defined in (2).

**Remark.** It is known (see, for example, [1] and [2]) that if  $\varphi''(x) \geq C_0 > 0$  for all  $x \in \mathbb{R}$  then  $C(n, C_0, 0) = 4n^2/C_0$ . Since we need merely the existence of some constant  $C(n, C_0, R)$ , we can relax the assumption on  $\varphi$  and require the convexity of  $\varphi$  only for large  $|x|$ . (See p. 201-202 of [5] on how to obtain one logarithmic Sobolev inequality directly from another.)

Let  $\mu_i \stackrel{\text{def}}{=} \mu_{N, y_N}(\cdot | \mathcal{G}_i)$  and  $\tilde{\mu}_i$  be the marginal of  $\mu_i$  on the  $i$ -th box. Then  $f_i = E_{\mu_i} f_{N, y_N} = E_{\tilde{\mu}_i} f_{i-1}$ . It is easy to see that  $\tilde{\mu}_i$  is exactly in the form described in Theorem 4.1 ( $n = |B_i|$ ,  $y = \bar{x}_i$ ). Let

$$\partial_{j-1, j} = \left( \frac{\partial}{\partial x_{j-1}} - \frac{\partial}{\partial x_j} \right).$$

Setting  $\tilde{f}_i^s = f_{i-1}^s / f_i^s$  and applying (26) with  $f = \tilde{f}_i^s$  we obtain

$$\begin{aligned} E_{\tilde{\mu}_i} \tilde{f}_i^s \log \tilde{f}_i^s &\leq C(l, C_0, R) \int_{\mathbb{X}_{l, \bar{x}_i}} \sum_{j-1, j \in B_i} \left( \partial_{j-1, j} \sqrt{\tilde{f}_i^s} \right)^2 d\tilde{\mu}_i \\ &= \frac{C(l, C_0, R)}{4f_i^s} \int_{\mathbb{X}_{l, \bar{x}_i}} \sum_{j-1, j \in B_i} \frac{(\partial_{j-1, j} f_{i-1}^s)^2}{f_{i-1}^s} d\tilde{\mu}_i. \end{aligned} \quad (27)$$

Moreover,

$$\partial_{j-1, j} f_{i-1}^s = \partial_{j-1, j} E_{\mu_{i-1}} f_{N, y_N}^s = E_{\mu_{i-1}} \partial_{j-1, j} f_{N, y_N}^s, \quad j-1, j \in B_i. \quad (28)$$

The term where  $\partial_{j-1, j}$  is applied to  $\mu_{i-1}$  vanishes because  $\mu_{i-1}$  depends only on the sum of  $x_j$ ,  $j \in \cup_{n=i}^{k_N} B_n$ , and  $\partial_{j-1, j}$  applied to this sum is equal to zero. Using (28) and Hölder's we find that

$$\begin{aligned} \int_{\mathbb{X}_{l, \bar{x}_i}} \sum_{j-1, j \in B_i} \frac{(\partial_{j-1, j} f_{i-1}^s)^2}{f_{i-1}^s} d\tilde{\mu}_i &= \int_{\mathbb{X}_{l, \bar{x}_i}} \sum_{j-1, j \in B_i} \frac{(E_{\mu_{i-1}} \partial_{j-1, j} f_{N, y_N}^s)^2}{f_{i-1}^s} d\tilde{\mu}_i \\ &\leq \int_{\mathbb{X}_{l, \bar{x}_i}} 4 \sum_{j-1, j \in B_i} E_{\mu_{i-1}} \left( \partial_{j-1, j} \sqrt{f_{N, y_N}^s} \right)^2 d\tilde{\mu}_i \\ &= 4E_{\mu_i} \sum_{j-1, j \in B_i} \left( \partial_{j-1, j} \sqrt{f_{N, y_N}^s} \right)^2. \end{aligned} \quad (29)$$

From (29), (27), and (25) we obtain

$$\begin{aligned} \frac{1}{N} H(\nu_{N,y_N}^t \mid \mu_{N,y_N}) &\leq \frac{C(l, C_0, R)}{N^3(t-t_0)} \int_{t_0}^t D_{N,y_N}(\sqrt{f_{N,y_N}^s}) ds \\ &\quad + \frac{1}{N(t-t_0)} \int_{t_0}^t E_{\mu_{N,y_N}} f_k^s \log f_k^s ds, \end{aligned}$$

where the Dirichlet form  $D_{N,y_N}(\cdot)$  is given by (9).

The inequality (10) and the entropy bound (12) imply the following estimate on time averages of the Dirichlet form:

$$\int_{t_0}^t D_{N,y_N}(\sqrt{f_{N,y_N}^s}) ds = H(\nu_{N,y_N}^{t_0} \mid \mu_{N,y_N}) - H(\nu_{N,y_N}^t \mid \mu_{N,y_N}) \leq C(t_0)N.$$

This allows us to conclude that the first term in the right-hand side of (25) converges to zero as  $N \rightarrow \infty$ .

**Step 3.** We show that the last term in (25) admits the desired upper bound.

Denote by  $\bar{\nu}_{k_N}^s(\bar{\mu}_{k_N})$  the joint distribution of averages  $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{k_N}$  under  $\nu_{N,y_N}^t(\mu_{N,y_N})$  respectively. We have  $d\bar{\nu}_{k_N}^s = f_{k_N}^s d\bar{\mu}_{k_N}$ , and

$$E_{\mu_{N,y_N}} f_{k_N}^s \log f_{k_N}^s = E_{\nu_{N,y_N}^s} \log \frac{d\bar{\nu}_{k_N}^s}{d\sigma_{k_N}} - E_{\nu_{N,y_N}^s} \log \frac{d\bar{\mu}_{k_N}}{d\sigma_{k_N,y_N}}, \quad (30)$$

where  $\sigma_{k_N,y_N}$  is the Lebesgue measure on  $\mathbb{X}_{k_N,y_N}$ .

In Section 5 we prove

**Lemma 4.2.** *Under the assumptions of Theorem 2.2 for any  $t > t_0 > 0$*

$$\limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N(t-t_0)} \int_{t_0}^t E_{\nu_{N,y_N}^s} \log \frac{d\bar{\nu}_{k_N}^s}{d\sigma_{k_N,y_N}} ds \leq 0.$$

To finish the proof of Theorem 2.2 we only have to deal with the last term in (30).

For  $n \in \mathbb{N}$  define

$$h_n(y) \stackrel{\text{def}}{=} -\frac{1}{n} \log \int_{\mathbb{X}_{n,y}} \exp\left(-\sum_{j=1}^n \varphi(x_j)\right) d\sigma_{n,y}.$$

Then

$$\begin{aligned} \frac{1}{N} \log \frac{d\bar{\mu}_{k_N}}{d\sigma_{k_N, y_N}} &= -\frac{1}{N} \sum_{i=1}^{k_N} |B_i| h_{|B_i|}(\bar{x}_i) \\ &\quad - \frac{1}{N} \log \int_{\mathbb{X}_{N, y_N}} \exp\left(-\sum_{i \in \Lambda_N} \varphi(x_i)\right) d\sigma_{N, y_N}. \end{aligned}$$

By Theorem A.1 we know that

$$\lim_{n \rightarrow \infty} h_n(y) = h(y) \quad \text{locally uniformly in } y,$$

and, since  $\lim_{N \rightarrow \infty} y_N = \bar{m}$ , we have that

$$\lim_{N \rightarrow \infty} h_N(y_N) = h(\bar{m}).$$

Therefore, we arrive at the estimate

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{N} H(\nu_{N, y_N}^t | \mu_{N, y_N}) &\leq \\ \limsup_{N \rightarrow \infty} \frac{1}{N(t-t_0)} \int_{t_0}^t E_{\nu_{N, y_N}^s} \sum_{i=1}^{k_N} |B_i| h_{|B_i|}(\bar{x}_i) ds &- h(\bar{m}) \quad (31) \end{aligned}$$

for any  $l \in \mathbb{N}$  and  $t_0 \in (0, t)$ .

**Step 4.** For  $j \in \Lambda_N$  define the shift operator  $T_j$  on  $\mathbb{R}^N$  and also on functions  $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$  by

$$T_j x_i = x_{i+j}, \quad i \in \Lambda_N,$$

and

$$T_j f(x) = f(T_j x).$$

We also set for any  $n \in \mathbb{N}$

$$\bar{x}_{i,n} = \frac{1}{2n+1} \sum_{|j-i| \leq n} x_j. \quad (32)$$

Notice that the left-hand side of (31) does not depend on the way we divide the circle  $S$  into boxes. This observation implies that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} H(\nu_{N, y_N}^t | \mu_{N, y_N}) \leq \tag{33}$$

$$\limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N(t-t_0)} \int_{t_0}^t E_{\nu_{N, y_N}^s} \frac{1}{2l+1} \sum_{j=0}^{2l} \sum_{i=1}^{k_N} |B_i| h_{|B_i|}(T_j \bar{x}_i) ds - h(\bar{m}).$$

**Remark.** If  $N$  is divisible by  $(2l+1)$  then, clearly,  $|B_i| = 2l+1$  and

$$\frac{1}{N} E_{\nu_{N, y_N}^s} \frac{1}{2l+1} \sum_{j=0}^{2l} \sum_{i=1}^{k_N} |B_i| h_{|B_i|}(T_j \bar{x}_i) ds = \frac{1}{N} E_{\nu_{N, y_N}^s} \sum_{i \in \Lambda_N} h_{2l+1}(\bar{x}_{i,l}).$$

Otherwise there is a difference between the left-hand side and right-hand side in the above expression, which can not be a priori neglected even in the limit (as  $N \rightarrow \infty$  and then  $l \rightarrow \infty$ ), since functions  $h_n$  are neither bounded nor uniformly continuous.

Next we obtain an estimate on  $h_n$ , which is uniform in  $n$ .

**Lemma 4.3.** *Let  $\varphi \in C^2(\mathbb{R})$  satisfy  $(H_1)$ , (2), and (4) with  $p = 1$ . Then there is a constant  $M > 0$  such that*

$$|h_n(x)| \leq M(|\varphi(x)| + 1), \quad x \in \mathbb{R},$$

*uniformly in  $n$ .*

The proof of this lemma is given in Section 6.

The next lemma combined with Lemma 4.3 provides us with the uniform integrability, which we need when passing from the convergence in probability (relative to  $\nu_{N, y_N}^s$ ) to the convergence in  $L^1(\mathbb{X}_{N, y_N}, d\nu_{N, y_N}^s)$ .

**Lemma 4.4.** *Under the assumptions of Theorem 2.2 there is  $q > 1$  and a constant  $C$  such that*

$$E_{\nu_{N, y_N}^s} \left( \frac{1}{N} \sum_{i \in \Lambda_N} |\varphi(x_i)| \right)^q \leq C$$

*uniformly in  $N$  and locally uniformly in  $s > 0$ .*

We postpone the proof of this lemma until Section 6.

We use Lemma 4.3 and Lemma 4.4 to justify the following replacements. At first we replace the expression under the integral sign in (33) with

$$E_{\nu_{N,y_N}^s} \sum_{i \in \Lambda_N} h(\bar{x}_{i,l}), \quad (34)$$

where  $\bar{x}_{i,l}$  is defined by (32) with  $n = l$ . Then we substitute the averages over small macroscopic blocks  $\bar{x}_{i,\varepsilon_N}$  for the averages over large microscopic blocks  $\bar{x}_{i,l}$  obtaining

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{N} H(\nu_{N,y_N}^t | \mu_{N,y_N}) &\leq \\ \lim_{\varepsilon \rightarrow 0} \limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{(t-t_0)} \int_{t_0}^t \frac{1}{N} E_{\nu_{N,y_N}^s} h(\bar{x}_{i,\varepsilon_N}) ds - h(\bar{m}). \end{aligned} \quad (35)$$

This substitution is based on the following two-block estimate.

**Theorem 4.2 (two-block estimate).** *Under the assumptions of Theorem 2.1 for any  $t_2 > t_1 > 0$*

$$\lim_{\varepsilon \rightarrow 0} \limsup_{l \rightarrow \infty} \limsup_{N \rightarrow \infty} \int_{t_1}^{t_2} E_{\nu_{N,y_N}^s} \frac{1}{N} \sum_{i \in \Lambda_N} |\bar{x}_{i,l} - \bar{x}_{i,\varepsilon_N}| ds = 0.$$

This theorem is an immediate consequence of the entropy estimate (12) and Theorem 4.7 of [3].

The existence of the scaling limit implies that for any  $u \in S$ ,  $\varepsilon > 0$ , and  $\delta > 0$

$$\nu_{N,y_N}^s \left\{ x \in \mathbb{X}_{N,y_N} : \left| \bar{x}_{[uN],\varepsilon_N} - \frac{1}{2\varepsilon} \int_{u-\varepsilon}^{u+\varepsilon} m(s,\theta) d\theta \right| > \delta \right\} \rightarrow 0$$

as  $N \rightarrow \infty$ . Thus we can replace the expression under the integral sign in (35) with

$$\frac{1}{N} \sum_{i \in \Lambda_N} h \left( \frac{1}{2\varepsilon} \int_{i/N-\varepsilon}^{i/N+\varepsilon} m(s,\theta) d\theta \right). \quad (36)$$

But (36) is the integral sum for

$$\int_S h \left( \frac{1}{2\varepsilon} \int_{u-\varepsilon}^{u+\varepsilon} m(s,\theta) d\theta \right) du.$$

Then, since  $h$  and  $m$  are continuous functions, we can pass to the limit as  $\varepsilon \rightarrow 0$ . Finally we let  $t_0 \rightarrow t$  and obtain the upper bound (24).

## 5 Proof of Lemma 4.2.

For this Lemma we need, in fact, only assumptions  $(H_1)$ ,  $(H'_3)$ , the inequality (3) and the entropy estimate (12).

For each  $N$  and  $s \in [t_0, t]$  we introduce a new process  $x_{s_N}^{(N)}(\cdot)$ , which satisfies (1) up to time  $s_N < s$ . At time  $s_N$  we “switch off” the drift, and then the process  $x_{s_N}^{(N)}$  evolves as a Gaussian diffusion process with constant covariance. For all quantities associated with the new process we keep the same notation as for  $x^{(N)}$  but add a subscript  $s_N$ . In particular, the corresponding measure on the space of continuous paths will be denoted by  $P_{s_N}^{(N)}$ .

We have

$$\begin{aligned} \frac{1}{(t-t_0)N} \int_{t_0}^t E_{\nu_{N,y_N}^s} \log \frac{d\bar{\nu}_{k_N}^s}{d\sigma_{k_N,y_N}} ds &= \frac{1}{(t-t_0)N} \int_{t_0}^t E_{\nu_{N,y_N}^s} \log \frac{d\bar{\nu}_{k_N}^s}{d\nu_{s_N,k_N}^s} ds \\ &+ \frac{1}{(t-t_0)N} \int_{t_0}^t E_{\nu_{N,y_N}^s} \log \frac{d\nu_{s_N,k_N}^s}{d\sigma_{k_N,y_N}} ds. \end{aligned} \quad (37)$$

The idea of the proof is to choose the “switch off” time  $s_N$  in such a way that both terms in the right-hand side of (37) can be bounded above by quantities, which vanish when we pass to the limit as  $N \rightarrow \infty$  and then as  $l \rightarrow \infty$ . It turns out that the optimal choice for  $s_N$  is given by  $s_N = s - k_N/N^3$ , where  $k_N = [N/(2l+1)]$ .

We rewrite the system (1) using matrix notation.

$$dx(t) = N^2 A^{(N)} b^{(N)}(x(t)) dt + N G^{(N)} d\beta^{(N)},$$

where  $\beta^{(N)}$  is a standard  $N$ -dimensional Brownian motion,  $A^{(N)} = G^{(N)} G^{(N)*}$  and

$$G^{(N)} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \\ -1 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad b^{(N)}(x) = -\frac{1}{2} \begin{pmatrix} \varphi'(x_1) \\ \varphi'(x_2) \\ \dots \\ \dots \\ \varphi'(x_N) \end{pmatrix} \quad (38)$$

We now give a formal description of  $x_{s_N}^{(N)}$ . The process  $x_{s_N}^{(N)}$  is defined by the system

$$dx(t) = N^2 A^{(N)} b_{s_N}^{(N)}(t, x(t)) dt + N G^{(N)} d\beta^{(N)}(t)$$

where

$$b_{s_N}^{(N)}(t, \cdot) = \begin{cases} b^{(N)}(\cdot) & \text{if } t \leq s_N, \\ 0 & \text{if } t > s_N, \end{cases} \quad (39)$$

and  $G^{(N)}$ ,  $b^{(N)}(x)$  are given by (38). Assume that  $x_{s_N}^{(N)}(0) = \eta^{(N)}$ .

To estimate the first term in the right-hand side of (37) we notice that by the convexity of function  $x \log x$  and Jensen's inequality

$$E_{\nu_{N, y_N}^s} \log \frac{d\bar{\nu}_{k_N}^s}{d\nu_{s_N, k_N}^s} \leq E_{P^{(N)}} \log \frac{dP^{(N)}}{dP_{s_N}^{(N)}} \Big|_{\mathcal{F}_s},$$

where  $\mathcal{F}_s$  is the  $\sigma$ -algebra up to time  $s$ .

**Proposition 5.1.**

$$\frac{1}{N} E_{P^{(N)}} \log \frac{dP^{(N)}}{dP_{s_N}^{(N)}} \Big|_{\mathcal{F}_s} = \frac{N}{8} E_{P^{(N)}} \int_{s_N}^s (\varphi'(x_i(\tau)) - \varphi'(x_{i+1}(\tau)))^2 d\tau \quad (40)$$

**Proof.** By Girsanov's formula

$$\begin{aligned} \log \frac{dP_{s_N}^{(N)}}{dP^{(N)}} \Big|_{\mathcal{F}_s} &= \\ &- \int_0^s (b^{(N)} - b_{s_N}^{(N)}) d\bar{x} - \frac{N^2}{2} \int_0^s \langle b^{(N)} - b_{s_N}^{(N)}, A^{(N)}(b^{(N)} - b_{s_N}^{(N)}) \rangle d\tau \end{aligned}$$

where  $\bar{x} = x - N^2 \int_0^s A^{(N)} b^{(N)} d\tau$  is a  $P^{(N)}$ -martingale and  $\langle \cdot, \cdot \rangle$  denotes a scalar product in  $\mathbb{R}^N$ . Since  $b_{s_N}^{(N)}$  coincides with  $b$  up to time  $s_N$  and vanishes for all  $\tau > s_N$  we obtain

$$\log \frac{dP^{(N)}}{dP_{s_N}^{(N)}} \Big|_{\mathcal{F}_s} = \int_{s_N}^s b^{(N)} d\bar{x} + \frac{N^2}{2} \int_{s_N}^s \langle b^{(N)}, A^{(N)} b^{(N)} \rangle d\tau$$

Taking expectation with respect to  $P^{(N)}$  in the above equality and using (38) we obtain (40).  $\square$

**Proposition 5.2.** *Let  $s_N = s - \varepsilon_N > t_0/2$  and  $\varphi$  satisfies (3). Assume also that the entropy estimate (12) holds. Then*

$$\frac{1}{(t - t_0)} E_{P^{(N)}} \int_{t_0}^t \int_{s_N}^s (\varphi'(x_i(\tau)) - \varphi'(x_{i+1}(\tau)))^2 d\tau ds \leq NC(t, t_0) \varepsilon_N.$$

**Proof.** *Step 1.* Applying Itô's formula to

$$\sum_{i \in \Lambda_N} \varphi(x_i(s))$$

and using (1) we find that for any  $t_2 > t_1 > 0$

$$\frac{1}{2} E_{P^{(N)}} \int_{t_1}^{t_2} \sum_{i \in \Lambda_N} (\varphi'(x_i(s)) - \varphi'(x_{i+1}(s)))^2 ds = \quad (41)$$

$$\frac{1}{N^2} E_{P^{(N)}} \left( \sum_{i \in \Lambda_N} \varphi(x_i(t_1)) - \sum_{i \in \Lambda_N} \varphi(x_i(t_2)) \right) + E_{P^{(N)}} \int_{t_1}^{t_2} \sum_{i \in \Lambda_N} \varphi''(x_i(s)) ds.$$

Since we assumed that  $\varphi''(x) \leq C_1 \varphi(x) + C_2$ , the last term in the above equality is bounded by

$$C_1 E_{P^{(N)}} \int_{t_1}^{t_2} \sum_{i \in \Lambda_N} \varphi(x_i(s)) ds + C_2 N(t_2 - t_1).$$

*Step 2.* We show that for any  $t_0 > 0$  there is a constant  $C = C(t_0)$  such that for all  $N$  and all  $t > t_0$

$$\frac{1}{N} E_{\nu_{N,y_N}^t} \sum_{i \in \Lambda_N} \varphi(x_i) \leq C(t_0). \quad (42)$$

By the entropy inequality and (12) for any  $\delta \in (0, 1)$

$$\frac{1}{N} E_{\nu_{N,y_N}^t} \sum_{i \in \Lambda_N} \varphi(x_i) \leq \frac{1}{N\delta} \log E_{\mu_{N,y_N}} e^{\delta \sum_{i \in \Lambda_N} \varphi(x_i)} + \frac{1}{\delta} C(t_0). \quad (43)$$

By Theorem A.1 the first term in the right-hand side of (43) converges to a limit as  $N \rightarrow \infty$ . The inequality (42) follows.

*Step 3.* Combining the results of Steps 1 and 2 we conclude that

$$\begin{aligned} & \frac{1}{(t - t_0)} E_{P^{(N)}} \int_{t_0}^t \int_{s_N}^s (\varphi'(x_i(\tau)) - \varphi'(x_{i+1}(\tau)))^2 d\tau ds \leq \\ & \frac{2}{N^2(t - t_0)} E_{P^{(N)}} \int_{t_0}^t \left( \sum_{i \in \Lambda_N} \varphi(x_i(s_N)) - \sum_{i \in \Lambda_N} \varphi(x_i(s)) \right) ds + \\ & \quad + NC(t_0)\varepsilon_N. \quad (44) \end{aligned}$$

Notice that

$$\begin{aligned}
& E_{P^{(N)}} \int_{t_0}^t \left( \sum_{i \in \Lambda_N} \varphi(x_i(s_N)) - \sum_{i \in \Lambda_N} \varphi(x_i(s)) \right) ds = \\
& E_{P^{(N)}} \int_{t_0 - \varepsilon_N}^{t_0} \sum_{i \in \Lambda_N} \varphi(x_i(s)) ds - E_{P^{(N)}} \int_{t - \varepsilon_N}^t \sum_{i \in \Lambda_N} \varphi(x_i(s)) ds \leq NC(t_0)\varepsilon_N, \quad (45)
\end{aligned}$$

where the last inequality follows from Step 2. Relations (44) and (45) imply the statement of Proposition 5.2.  $\square$

Propositions 5.1 and 5.2 allow us to conclude that for  $s_N = s - \varepsilon_N > t_0/2$

$$\frac{1}{N(t - t_0)} \int_{t_0}^t E_{\nu_{N, y_N}^s} \log \frac{d\bar{\nu}_{k_N}^s}{d\nu_{s_N, k_N}^s} \leq N^2 C(t, t_0) \varepsilon_N.$$

In particular, for  $\varepsilon_N = k_N/N^3$ , where  $k_N = [N/(2l + 1)]$ , the right-hand side of the above inequality converges to zero when we let  $N \rightarrow \infty$  and then  $l \rightarrow \infty$ .

We turn now to the last term in (37). After time  $s_N$  the box averages  $(\bar{x}_{s_N, 1}, \bar{x}_{s_N, 2}, \dots, \bar{x}_{s_N, k_N})$  undergo a Gaussian diffusion in  $\mathbb{R}^{k_N}$  with parameters  $(0, k_N^2 A^{(k_N)})$ , where  $A^{(k_N)} : \mathbb{R}^{k_N} \rightarrow \mathbb{R}^{k_N}$  is given by

$$A^{(k_N)} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}.$$

Let  $\tilde{A}^{(k_N)}$  be the restriction of  $A^{(k_N)}$  to  $\mathbb{X}_{k_N, y_N}$ . Then  $\tilde{A}^{(k_N)}$  is non-degenerate and

$$\begin{aligned}
\frac{1}{N} \log \frac{d\nu_{s_N, k_N}^s}{d\sigma_{k_N, y_N}}(\cdot) &= \frac{1}{N} \log \left[ \left( (2\pi k_N^2 \varepsilon_N)^{k_N - 1} \det \tilde{A}^{(k_N)} \right)^{-1/2} \times \right. \\
&\quad \left. \int_{\mathbb{X}_{k_N, y_N}} \exp \left( -(x - \cdot)^T \tilde{A}^{(k_N)^{-1}} (x - \cdot) / (2\varepsilon_N) \right) d\nu_{s_N, k_N}^s \right] \leq \\
&\quad - \frac{k_N - 1}{2N} \log (2\pi k_N^2 \varepsilon_N) - \frac{1}{2N} \log \det \tilde{A}^{(k_N)}. \quad (46)
\end{aligned}$$

The  $\det \tilde{A}^{(k_N)}$  is equal to the product of non-zero eigenvalues of  $A^{(k_N)}$ .

**Proposition 5.3.** For  $k = 3, 4, \dots$  we have

$$\det \tilde{A}^{(k)} = 2^{k-1} \prod_{j=1}^{k-1} \left(1 - \cos \frac{2\pi j}{k}\right);$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log(\det \tilde{A}^{(k)}) = 2 \log 2 + 4 \int_0^{\frac{1}{2}} \log \sin(\pi x) dx.$$

**Proof.** Denote  $2\pi j/k$  by  $\alpha_j^{(k)}$ . It is easy to check that vectors  $e_j^{(k)} = (e^{i\alpha_j^{(k)}}, \dots, e^{ik\alpha_j^{(k)}})$ ,  $j = 0, \dots, k-1$ , are independent eigenvectors for  $A^{(k)}$  with eigenvalues  $2(1 - \cos \alpha_j^{(k)})$  respectively. Taking the product of non-zero eigenvalues ( $j = 1, \dots, k-1$ ) we obtain the first part of the proposition.

To prove the second statement notice that

$$\begin{aligned} \frac{1}{k} \log \prod_{j=1}^{k-1} \left(1 - \cos \frac{2\pi j}{k}\right) &= \frac{k-1}{k} \log 2 + \sum_{i=1}^{k-1} \frac{1}{k} \log \sin^2 \frac{\pi j}{k} \\ &= \frac{k-1}{k} \log 2 + 4 \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{k} \log \sin \frac{\pi j}{k}. \end{aligned} \quad (47)$$

The last sum is the integral sum for the convergent integral

$$\int_0^{\frac{1}{2}} \log \sin(\pi x) dx.$$

Passing to the limit as  $k \rightarrow \infty$  concludes the proof.  $\square$

Let  $s - s_N = k_N/N^3$ . Since  $k_N = \lfloor N/(2l+1) \rfloor$ , we obtain from (46) and Proposition 5.3 that

$$\lim_{l \rightarrow \infty} \lim_{N \rightarrow \infty} \left( -\frac{k_N - 1}{2N} \log(2\pi k_N^2 \varepsilon_N) - \frac{1}{2N} \log \det \tilde{A}^{(k_N)} \right) = 0.$$

The proof of Lemma 4.2 is complete.

## 6 Proofs of Lemma 4.3 and Lemma 4.4.

**Proof of Lemma 4.3.** It is clear that for any  $x' < x''$

$$\begin{aligned} \int_{x'}^{x''} e^{-h_n(x)} dx &= \int_{nx' \leq \sum_{j=1}^n x_j \leq nx''} e^{-\sum_{j=1}^n \varphi(x_j)} d^n x \\ &\geq \int_{\substack{x' \leq x_j \leq x'' \\ j=1, \dots, n}} e^{-\sum_{j=1}^n \varphi(x_j)} d^n x = \left( \int_{x'}^{x''} e^{-\varphi(x)} dx \right)^n \end{aligned}$$

If  $h_n$  and  $\varphi$  are monotone increasing for  $x > x_0 > 0$  then

$$(x'' - x') \exp(-nh_n(x')) \geq (x'' - x')^n \exp(-n\varphi(x'')) \quad \text{for } x', x'' \in [x_0, \infty),$$

i. e.  $h_n(x') \leq -(1 - \frac{1}{n}) \log(x'' - x') + \varphi(x'')$ , which implies

$$h_n(x) \leq \varphi(x + 1) \quad \text{for } x > x_0. \quad (48)$$

Similarly if  $h_n$  and  $\varphi$  are monotone decreasing for  $x < -x_0$  then

$$h_n(x) \leq \varphi(x - 1) \quad \text{for } x < -x_0. \quad (49)$$

Since  $\varphi$  is convex for large  $|x|$  and  $\int e^{-\varphi(x)} dx < \infty$  we know that it is monotone increasing for large positive  $x$  and is monotone decreasing for large negative  $x$ . Moreover,

$$\begin{aligned} \varphi'(x) &\geq C_0 x - C & \text{for } x > x_0 & \text{ and} \\ \varphi'(x) &\leq C_0 x + C & \text{for } x < -x_0 \end{aligned}$$

for some  $x_0 > 0$ . These inequalities hold also for  $h'_n$ . Indeed, if  $x > x_0$  then

$$h'_n(x) = \frac{\int_{\mathbb{X}_{n,x}} \frac{1}{n} \sum_{j=1}^n \varphi'(x_j) \exp\left(-\sum_{j=1}^n \varphi(x_j)\right) d\sigma_{n,x}}{\int_{\mathbb{X}_{n,x}} \exp\left(-\sum_{j=1}^n \varphi(x_j)\right) d\sigma_{n,x}} \geq C_0 x - C.$$

Similarly,  $h'_n(x) \leq C_0 x + C$  for  $x < -x_0$ . This implies monotonicity of  $h_n$  for large  $|x|$ . Using the condition on the derivative of  $\varphi$  we find that

$$|\varphi(x \pm 1)| \leq C(|\varphi(x)| + 1) \quad (50)$$

for some constant  $C > 0$ . The statement of the lemma now follows from (48) - (50) by noticing that  $h_n$  are bounded below uniformly in  $n$ . The last observation is a consequence of monotonicity of  $h_n$  for large  $|x|$  and Theorem A.1.  $\square$

**Proof of Lemma 4.4.** Let  $q = \min\{2, 1 + \delta_0/2p\}$ . Without loss of generality we can assume that  $\varphi \geq 0$ . Let

$$H_N^{(q)}(s) = \int_{\mathbb{X}_{N,y_N}} f_{N,y_N}^s |\log f_{N,y_N}^s|^q d\mu_{N,y_N}.$$

By Lemma B.1 we have

$$\begin{aligned} E_{\nu_{N,y_N}^s} \left( \frac{1}{N} \sum_{i \in \Lambda_N} \varphi(x_i) \right)^q &\leq \frac{C_q}{N^q} + \frac{4}{N^q} \log^q \left( 1 + E_{\mu_{N,y_N}} e^{\sum_{i \in \Lambda_N} \varphi(x_i)} \right) \\ &\quad + \frac{2}{N^q} H_N^{(q)}(s). \end{aligned} \quad (51)$$

The second term in the right-hand side of (51) is clearly bounded since

$$\frac{1}{N} \log E_{\mu_{N,y_N}} e^{\sum_{i \in \Lambda_N} \varphi(x_i)}$$

converges to a limit as  $N \rightarrow \infty$ . Thus we have reduced the problem to proving that for our choice of  $q$

$$H_N^{(q)}(s) \leq C_q N^q \quad (52)$$

locally uniformly in  $s > 0$ .

The proof of (52) is similar to the proof of (12) given in [4]. Even though the function  $x |\log x|^q$  is convex only for  $x \geq \exp(1 - q)$  it will not affect our considerations. There exists a smooth convex function  $K(x)$  such that

$$x |\log x|^q \leq K(x) \leq B + x |\log x|^q, \quad x \geq 0 \quad (53)$$

for some  $B > 0$ . We have

$$\begin{aligned} \frac{d}{ds} \int_{\mathbb{X}_{N,y_N}} K(f_{N,y_N}^s) d\mu_{N,y_N} &= \int_{\mathbb{X}_{N,y_N}} K'(f_{N,y_N}^s) \mathcal{L}_N f_{N,y_N}^s d\mu_{N,y_N} \\ &= - \int_{\mathbb{X}_{N,y_N}} K''(f_{N,y_N}^s) \sum_{i \in \Lambda_N} \left( \frac{\partial f_{N,y_N}^s}{\partial x_i} - \frac{\partial f_{N,y_N}^s}{\partial x_{i+1}} \right)^2 d\mu_{N,y_N} \leq 0 \end{aligned}$$

and, therefore, if (52) holds for some  $s_0 > 0$  with  $C_q = C_q(s_0)$  then it holds for all  $s \geq s_0$  with  $C_q \leq C_q(s_0) + B$ . In the next step we argue that if we let  $s_0$  depend on  $N$  in such a way that  $s_0(N) \rightarrow 0$  as  $N \rightarrow \infty$  and  $s_0(N)N^2$  is bounded below by some  $\tau > 0$  then the sequence  $H_N^{(q)}(s_0(N))/N^q$  remains bounded by a constant  $C = C(\tau)$ . This will finish the proof of (52).

*Step 2.* Let  $s_N = \tau N^{-2}$ ,  $\tau > 0$ , and define a non-speeded process  $z^{(N)}(\tau) = x^{(N)}(s_N)$ . Then  $z^{(N)}(\tau)$  satisfies the system

$$dz_i(\tau) = \frac{1}{2} (\varphi'(z_{i-1}) - 2\varphi(z_i) + \varphi(z_{i+1})) d\tau + d\beta_i(\tau) - d\beta_{i+1}(\tau), \quad (54)$$

with initial conditions  $z_i(0) = \eta_i^{(N)}$ ,  $i \in \Lambda_N$ .

By abuse of notation we write  $f_{N,y_N}^\tau$ ,  $\nu_{N,y_N}^\tau$ ,  $H_N^{(q)}(\tau)$  instead of  $f_{N,y_N}^{s_N}$ ,  $\nu_{N,y_N}^{s_N}$ ,  $H_N^{(q)}(s_N)$ .

We need to show that

$$H_N^{(q)}(\tau) = \int_{\mathbb{X}_{N,y_N}} f_{N,y_N}^\tau |\log f_{N,y_N}^\tau|^q d\mu_{N,y_N} \leq C(\tau)N^q$$

uniformly in  $N$ . This is done by ‘‘comparison’’ of  $z^{(N)}$  with a Gaussian process  $w^{(N)}$  which solves

$$dw_i^{(N)}(\tau) = d\beta_i^{(N)}(\tau) - d\beta_{i+1}^{(N)}(\tau), \quad i \in \Lambda_N,$$

with the same initial data.

Denote by  $r_{N,y_N}^\tau$  the density of the finite dimensional distribution of  $w^{(N)}(\tau)$  with respect to  $\mu_{N,y_N}$ . Then

$$\begin{aligned} & \int_{\mathbb{X}_{N,y_N}} f_{N,y_N}^\tau |\log f_{N,y_N}^\tau|^q d\mu_{N,y_N} \leq \\ & \leq \int_{\{f_{N,y_N}^\tau \geq 1\}} f_{N,y_N}^\tau \left| \log \left( \frac{f_{N,y_N}^\tau}{r_{N,y_N}^\tau} \cdot r_{N,y_N}^\tau \right) \right|^q d\mu_{N,y_N} + \max_{0 \leq x \leq 1} x |\log x|^q. \end{aligned}$$

It is evident that for any  $0 < x \leq y$ ,  $xy \geq 1$

$$0 \leq \log(xy) \leq 2 \log y.$$

We have

$$\begin{aligned}
& \int_{\{f_{N,y_N}^\tau \geq 1\}} f_{N,y_N}^\tau \left| \log \left( \frac{f_{N,y_N}^\tau}{r_{N,y_N}^\tau} \cdot r_{N,y_N}^\tau \right) \right|^q d\mu_{N,y_N} \leq \\
& \leq 2^q \int_{\{f_{N,y_N}^\tau \geq 1\}} f_{N,y_N}^\tau \left( \max \left\{ \log \frac{f_{N,y_N}^\tau}{r_{N,y_N}^\tau}, \log r_{N,y_N}^\tau \right\} \right)^q d\mu_{N,y_N} \\
& \leq 2^q \left( \int_{\mathbb{X}_{N,y_N}} f_{N,y_N}^\tau \left| \log \frac{f_{N,y_N}^\tau}{r_{N,y_N}^\tau} \right|^q d\mu_{N,y_N} + \int_{\{r_{N,y_N}^\tau \geq 1\}} f_{N,y_N}^\tau \log^q r_{N,y_N}^\tau d\mu_{N,y_N} \right).
\end{aligned}$$

The first term of the sum above will be estimated by Girsanov's formula. At this point our assumption (20) on the initial data comes into play. The fact that  $r_{N,y_N}^\tau$  is known explicitly will allow us to obtain a bound on the second term. The following two propositions complete the proof of Lemma 4.4.

**Proposition 6.1.** *Assume that  $\{\eta^{(N)}\}$  satisfies (20) where  $p$  is defined in  $(H_3)$ . Then*

- i)  $E_{\nu_{N,y_N}^\tau} \sum_{i \in \Lambda_N} |\varphi(z_i^{(N)})|^{2pq} \leq C(T)N$  for all  $\tau \in [0, T]$ ,  $T < \infty$ ;
- ii)  $\int_{\mathbb{X}_{N,y_N}} f_{N,y_N}^\tau \left| \log \frac{f_{N,y_N}^\tau}{r_{N,y_N}^\tau} \right|^q d\mu_{N,y_N} \leq C(\tau)N^q$  for any  $\tau > 0$ .

**Proposition 6.2.** *For any  $\tau > 0$  there is a constant  $C(\tau)$  such that*

$$\int_{r_{N,y_N}^\tau \geq 1} f_{N,y_N}^\tau \log^q r_{N,y_N}^\tau d\mu_{N,y_N} \leq C(\tau)N^q.$$

**Proof of Proposition 6.1.** To simplify the notation we drop the superscript in  $z^{(N)}$ .

i) First we prove a slightly more general statement. Let  $\psi(x)$  be any smooth convex function which satisfies the inequality

$$\psi''(x) \leq C_1\psi(x) + C_2. \tag{55}$$

By Itô's formula

$$d\psi(z_i) = \psi''(z_i) d\tau + \psi'(z_i) dz_i.$$

Taking summation over  $i$ , integrating from 0 to  $\tau$ , applying (54) and computing the expectation we obtain

$$\begin{aligned}
E_{P_N} \sum_{i \in \Lambda_N} \psi(z_i(\tau)) &= \sum_{i \in \Lambda_N} \psi(\eta_i^{(N)}) + \int_0^\tau E_{P_N} \psi''(z_i(s)) ds \\
&\quad - \frac{1}{2} \int_0^\tau E_{P_N} \sum_{i \in \Lambda_N} (\psi'(z_{i+1}(s)) - \psi'(z_i(s))) (\varphi'(z_{i+1}(s)) - \varphi'(z_i(s))) ds \\
&\leq \sum_{i \in \Lambda_N} \psi(\eta_i^{(N)}) + C_2 N \tau + C_1 \int_0^\tau E_{P_N} \sum_{i \in \Lambda_N} \psi(z_i(s)) ds.
\end{aligned}$$

In the last inequality we used convexity of  $\varphi$  and  $\psi$  and the condition (55). An application of the Gronwall inequality yields

$$E_{\nu_{N,y_N}^\tau} \sum_{i \in \Lambda_N} \psi(z_i) \leq \left( \sum_{i \in \Lambda_N} \psi(\eta_i^{(N)}) + N(1 - e^{-C\tau}) \right) e^{C\tau}. \quad (56)$$

In our case we can take  $\psi(x)$  essentially to be equal to  $|\varphi(x)|^{2pq}$  modifying the latter on a finite interval as necessary. Then (56) implies that for  $\tau \in [0, T]$

$$E_{\nu_{N,y_N}^\tau} \sum_{i \in \Lambda_N} |\varphi(z_i)|^{2pq} \leq C(T)N.$$

*ii)* We show that  $\int_{\mathbb{X}_{N,y_N}} K \left( \frac{f_{N,y_N}^\tau}{r_{N,y_N}^\tau} \right) r_{N,y_N}^\tau d\mu_{N,y_N} \leq C(\tau)N^q$  where  $K$  satisfies (53). This immediately implies *ii)*. By convexity of  $K$

$$\begin{aligned}
\int_{\mathbb{X}_{N,y_N}} K \left( \frac{f_{N,y_N}^\tau}{r_{N,y_N}^\tau} \right) r_{N,y_N}^\tau d\mu_{N,y_N} &\leq E_{Q_N} K \left( \frac{dP_N}{dQ_N} \right)_{\mathcal{F}_\tau} \Big| \\
&\leq B + E_{P_N} \left| \log \frac{dP_N}{dQ_N} \right|_{\mathcal{F}_\tau} \Big|^q.
\end{aligned}$$

where  $Q_N$  is the measure on continuous paths associated with  $w^{(N)}$ . In the

same way as in the proof of Proposition 5.1 we obtain

$$\begin{aligned}
& E_{P_N} \left| \log \frac{dP_N}{dQ_N} \Big|_{\mathcal{F}_\tau} \right|^q = \\
& = E_{P_N} \left| \int_0^\tau b^{(N)}(z(s)) d\bar{z}(s) + \frac{1}{2} \int_0^\tau \langle b^{(N)}(z), A^{(N)}b^{(N)}(z) \rangle(s) ds \right|^q \\
& \leq 2^{q-1} \left| E_{P_N} \int_0^\tau b^{(N)}(z(s)) d\bar{z}(s) \right|^q + C(\tau) E_{P_N} \int_0^\tau \langle b^{(N)}(z), A^{(N)}b^{(N)}(z) \rangle^q(s) ds
\end{aligned}$$

with  $\bar{z}(s) = z(s) - \int_0^s A^{(N)}b^{(N)}(z(s)) ds$ .

From the inequality  $a^q \leq C(q)(a^2 + 1)$ , where  $q \in [1, 2]$ ,  $a > 0$ , and Itô's isometry we find that

$$\begin{aligned}
E_{P_N} \left| \log \frac{dP_N}{dQ_N} \Big|_{\mathcal{F}_\tau} \right|^q & \leq \tilde{C}(q) \left( E_{P_N} \int_0^\tau \langle b^{(N)}(z), A^{(N)}b^{(N)}(z) \rangle(s) ds + 1 \right) \\
& \quad + C(\tau) E_{P_N} \int_0^\tau \langle b^{(N)}(z(s)), A^{(N)}b^{(N)}(z(s)) \rangle^q ds \\
& < \tilde{C}(q, \tau) \left( E_{P_N} \int_0^\tau \langle b^{(N)}(z), A^{(N)}b^{(N)}(z) \rangle^q(s) ds + 1 \right).
\end{aligned}$$

Notice that by (38) and (4)

$$\begin{aligned}
\langle b^{(N)}(z), A^{(N)}b^{(N)}(z) \rangle & = \frac{1}{2} \sum_{i \in \Lambda_N} (\varphi'(z_{i+1}) - \varphi'(z_i))^2 \\
& \leq 4C_1^2 \left( \sum_{i \in \Lambda_N} |\varphi(z_i)|^{2p} + N \right).
\end{aligned}$$

Therefore by the Hölder inequality and part *i*)

$$\begin{aligned} E_{P_N} \int_0^\tau \langle b^{(N)}(z(s)), A^{(N)} b^{(N)}(z(s)) \rangle^q ds \\ \leq C \left( N^{q-1} E_{P_N} \int_0^\tau \sum_{i \in \Lambda_N} |\varphi(z_i(s))|^{2pq} ds + N^q \right) \leq C(\tau) N^q \end{aligned}$$

This completes the proof of Proposition 6.1.  $\square$

**Proof of Proposition 6.2.** Let  $\tilde{A}^{(N)}$  be the restriction of  $A^{(N)}$  to  $\mathbb{X}_{N,y_N}$ . Then by the definition of  $r_{N,y_N}^\tau$  for any  $z$  such that  $r_{N,y_N}^\tau(z) \geq 1$  we obtain

$$\begin{aligned} 0 \leq \frac{1}{N} \log r_{N,y_N}^\tau(z) &= -\frac{1}{N} \log \frac{d\mu_{N,y_N}}{d\sigma_{N,y_N}}(z) + \\ \frac{1}{N} \log \left[ ((2\pi\tau)^{N-1} \det \tilde{A}^{(N)})^{-\frac{1}{2}} \exp \left( -\frac{1}{2\tau} \langle z - \eta^{(N)}, (\tilde{A}^{(N)})^{-1} (z - \eta^{(N)}) \rangle \right) \right] \\ &\leq -\frac{N-1}{2N} \log(2\pi\tau) - \frac{1}{2N} \log(\det \tilde{A}^{(N)}) + \\ &\quad \frac{1}{N} \sum_{i \in \Lambda_N} \varphi(z_i) + \frac{1}{N} \log \int_{\mathbb{X}_{N,y_N}} e^{-\sum_{i \in \Lambda_N} \varphi(x_i)} d\sigma_{N,y_N}. \end{aligned}$$

The first term in the right-hand side is evidently bounded. The second and the fourth terms have finite limits as  $N \rightarrow \infty$  by Proposition 5.3 (with  $k = N$ ) and Theorem A.1 respectively. The third term is the only one which depends on  $z$ . We conclude that

$$\begin{aligned} \frac{1}{N^q} \int_{r_{N,y_N}^\tau \geq 1} f_{N,y_N}^\tau \log^q r_{N,y_N}^\tau d\mu_{N,y_N} &\leq C(\tau) + \frac{2^{q-1}}{N^q} \int_{\mathbb{X}_{N,y_N}} \left| \sum_{i \in \Lambda_N} \varphi(z_i) \right|^q d\nu_{N,y_N}^\tau \\ &\leq C(\tau) + \frac{2^{q-1}}{N^q} \int_{\mathbb{X}_{N,y_N}} N^{q-\frac{1}{2p}} \left( \sum_{i \in \Lambda_N} |\varphi(z_i)|^{2pq} \right)^{\frac{1}{2p}} d\nu_{N,y_N}^\tau \\ &= C(\tau) + 2^{q-1} \left( E_{\nu_{N,y_N}^\tau} \frac{1}{N} \sum_{i \in \Lambda_N} |\varphi(z_i)|^{2pq} \right)^{\frac{1}{2p}} \leq \tilde{C}(\tau) \end{aligned}$$

by part *i*) of Proposition 6.1.  $\square$

## 7 Proof of Theorem 2.3.

**Lemma 7.1.** *Under conditions of Theorem 2.2*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \int_{\mathbb{X}_{N,y_N}} f_{N,y_N}^t \log g_{N,y_N}^t d\mu_{N,y_N} = \int_0^1 h(m(t, \theta)) d\theta - h(\bar{m})$$

where  $\bar{m}$  is defined by (15).

**Proof.** Let  $m(t, \theta)$  be the solution of (19) with initial condition  $m_0$  and  $\lambda(t, \theta) = h'(m(t, \theta))$ . Denote  $\lambda(\frac{i}{N}, \theta)$  by  $\lambda_N^i$ . Recall that  $y_N = \frac{1}{N} \sum_{i \in \Lambda_N} \eta_i^{(N)}$ . The density  $g_{N,y_N}^t$  of a local Gibbs measure  $\gamma_N^{m_t}$  with respect to the invariant measure  $\mu_{N,y_N}$  is easily computed:

$$g_{N,y_N}^t(x) = \frac{\exp(\sum_{i \in \Lambda_N} \lambda_N^i x_i) \int_{\mathbb{X}_{N,y_N}} \exp(-\sum_{i \in \Lambda_N} \varphi(x_i)) d\sigma_{N,y_N}}{\int_{\mathbb{X}_{N,y_N}} \exp(\sum_{i \in \Lambda_N} \lambda_N^i x_i - \sum_{i \in \Lambda_N} \varphi(x_i)) d\sigma_{N,y_N}}, \quad x \in \mathbb{X}_{N,y_N}.$$

Hence

$$\begin{aligned} \frac{1}{N} \int_{\mathbb{X}_{N,y_N}} \log g_{N,y_N}^t d\nu_{N,y_N}^t &= \frac{1}{N} \int_{\mathbb{X}_{N,y_N}} \sum_{i \in \Lambda_N} \lambda_N^i x_i d\nu_{N,y_N}^t \\ &\quad - \frac{1}{N} \log \int_{\mathbb{X}_{N,y_N}} \exp\left(\sum_{i \in \Lambda_N} \lambda_N^i x_i - \sum_{i \in \Lambda_N} \varphi(x_i)\right) d\sigma_{N,y_N} \\ &\quad + \frac{1}{N} \log \int_{\mathbb{X}_{N,y_N}} \exp\left(-\sum_{i \in \Lambda_N} \varphi(x_i)\right) d\sigma_{N,y_N}. \end{aligned}$$

By Theorem A.1 and Theorem A.2 we obtain

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \log \int_{\mathbb{X}_{N,y_N}} \exp\left(-\sum_{i \in \Lambda_N} \varphi(x_i)\right) d\sigma_{N,y_N} &= -h(\bar{m}); \\ \lim_{N \rightarrow \infty} \frac{1}{N} \log \int_{\mathbb{X}_{N,y_N}} \exp\left(\sum_{i \in \Lambda_N} \lambda_N^i x_i - \sum_{i \in \Lambda_N} \varphi(x_i)\right) d\sigma_{N,y_N} \\ &= -\int_0^1 \log M(\lambda(t, \theta)) d\theta. \end{aligned}$$

The last equality follows from (18), (59) and the definition of  $\lambda$ . By the existence of the scaling limit (2.1) and Lemma 3.2 we have that

$$\lim_{N \rightarrow \infty} E_{\nu_{N,y_N}^t} \left| \frac{1}{N} \sum_{i \in \Lambda_N} \lambda_N^i x_i - \int_0^1 \lambda(t, \theta) m(t, \theta) d\theta \right| = 0. \quad (57)$$

This completes the proof of Lemma 7.1:

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \int_{\mathbb{X}_{N,y_N}} \log g_{N,y_N}^t d\nu_{N,y_N}^t &= \int_0^1 (\lambda m - \log M(\lambda))(t, \theta) d\theta - h(\bar{m}) \\ &= \int_0^1 h(m(t, \theta)) d\theta - h(\bar{m}). \end{aligned}$$

□

## A Limit theorems for densities.

Let  $\varphi$  satisfy  $(H_1)$ ,  $(H_2)$ , and  $(H'_3)$  and functions  $M$ ,  $\rho$ ,  $h$  be defined by  $((H'_3))$ , (16), (17) respectively.

The following theorem was proven in [3]:

**Theorem A.1.** *Let  $\varphi_N$  be the density of  $\frac{1}{N} \sum_{i \in \Lambda_N} X_i$ , where  $X_1, \dots, X_N$  are independent identically distributed random variables with the common density  $e^{-\varphi(x)}$ . Then*

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \log \varphi_N(x) &= -h(x), \\ \lim_{N \rightarrow \infty} \frac{1}{N} \frac{\varphi'_N(x)}{\varphi_N(x)} &= -h'(x) \end{aligned}$$

*uniformly on compact  $x$ -intervals.*

We prove a generalization of this theorem to the series of independent random variables  $X_N^1, X_N^2, \dots, X_N^N$ , which are not identically distributed but whose distributions vary slightly with  $i$ . More precisely, fix a function  $\lambda \in C([0, 1])$ . Denote  $\lambda(i/N)$  by  $\lambda_N^i$ , the sup-norm of  $\lambda$  by  $\|\lambda\|$ , and the set  $\{\lambda \in C([0, 1]) : \|\lambda\| \leq R\}$  by  $B_R^\lambda$ . Assume that  $X_N^i$  has the density

$$a(x, \lambda_N^i) = \frac{1}{M(\lambda_N^i)} \exp(\lambda_N^i x - \varphi(x)) \quad (58)$$

with respect to the Lebesgue measure,  $i = 1, 2, \dots, N$  and  $N \in \mathbb{N}$ .

**Theorem A.2.** Let  $X_N^1, X_N^2, \dots, X_N^N$  be defined as above and  $a_N^\lambda$  be the density of  $\sum_{i \in \Lambda_N} X_N^i / N$ . Then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log a_N^\lambda(x) = -I^\lambda(x)$$

uniformly on any set of the form  $\{|x| \leq R_1\} \times B_{R_2}^\lambda$ ,  $R_1, R_2 < \infty$ . The rate function  $I^\lambda$  is given by the relation

$$I^\lambda(x) = \sup_z \left\{ zx - \int_0^1 \log \frac{M(\lambda(\theta) + z)}{M(\lambda(\theta))} d\theta \right\}. \quad (59)$$

The proof of this theorem is based on the local limit theorem, which we now formulate. Define for any  $s \in \mathbb{R}$

$$b(y, s) = \frac{1}{M(s)} \exp(s(y + \rho'(s)) - \varphi(y + \rho'(s))). \quad (60)$$

Then it is easy to check that

$$\begin{aligned} \int b(y, s) dy &= 1, \quad \int y b(y, s) dy = 0 \\ \int y^2 b(y, s) dy &= \frac{M''(s)}{M(s)} - \left( \frac{M'(s)}{M(s)} \right)^2 = \rho''(s). \end{aligned}$$

**Theorem A.3.** Let  $Y_N^1, \dots, Y_N^N$  be independent random variables and the density of  $Y_N^i$  with respect to the Lebesgue measure is given by (60),  $i = 1, 2, \dots, N$ . Let  $b_N^\lambda$  be the density of  $\sum_{i \in \Lambda_N} Y_N^i / \sqrt{N}$ . Then for any non-negative integer  $k$  the functions  $b_N^\lambda$  belong to  $C^k(\mathbb{R})$  for all  $N \geq N_0(\|\lambda\|, k)$  and

$$\lim_{N \rightarrow \infty} \frac{d^k}{dy^k} b_N^\lambda(y) = \frac{1}{\sqrt{2\pi\sigma(\lambda)}} \frac{d^k}{dy^k} e^{-y^2/2\sigma^2(\lambda)}$$

uniformly on  $\mathbb{R} \times B_R^\lambda$ ,  $R < \infty$ . Here  $\sigma^2(\lambda) = \int_0^1 \rho''(\lambda(\theta)) d\theta$ .

The proof of Theorem A.3 is similar to the proof of Lemma 3.3 of [3], and is omitted.

**Proof of Theorem A.2.** From (58) and (60) we see that  $X_N^i$  has the same distribution as  $Y_N^i + \rho'(\lambda_N^i)$ . This implies the equivalence in law

$$\frac{1}{N} \sum_{i \in \Lambda_N} X_N^i \sim \frac{1}{\sqrt{N}} \left( \frac{1}{\sqrt{N}} \sum_{i \in \Lambda_N} Y_N^i \right) + \frac{1}{N} \sum_{i \in \Lambda_N} \rho'(\lambda_N^i),$$

from which we obtain that

$$a_N^\lambda(x) = \sqrt{N} b_N^\lambda \left( \sqrt{N} \left( x - \frac{1}{N} \sum_{i \in \Lambda_N} \rho'(\lambda_N^i) \right) \right).$$

By Theorem A.3 we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log a_N^\lambda \left( \frac{1}{N} \sum_{i \in \Lambda_N} \rho'(\lambda_N^i) \right) = \lim_{N \rightarrow \infty} \frac{1}{N} \left( \log \sqrt{N} + \log b_N^\lambda(0) \right) = 0. \quad (61)$$

Let  $z_0 = z_0(y)$  be the point where the supremum in (59) is attained and  $\tilde{\lambda}(\theta) = \lambda(\theta) + z_0$ . Introduce a new set of random variables  $\tilde{X}_N^i$ ,  $i = 1, \dots, N$ , with densities

$$\tilde{a}(x, \tilde{\lambda}_N^i) = \frac{1}{M(\tilde{\lambda}_N^i)} \exp \left( \tilde{\lambda}_N^i x - \varphi(x) \right).$$

Then by (61) the density  $a_N^{\tilde{\lambda}}$  of the average  $\sum_{i \in \Lambda_N} \tilde{X}_N^i / N$  satisfies

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log a_N^{\tilde{\lambda}} \left( \frac{1}{N} \sum_{i \in \Lambda_N} \rho'(\tilde{\lambda}_N^i) \right) = 0. \quad (62)$$

Since  $a(x, \lambda_N^i) = \tilde{a}(x, \tilde{\lambda}_N^i) e^{-z_0 x} M(\tilde{\lambda}_N^i) / M(\lambda_N^i)$ , using the properties of convolution we can compute that

$$a_N^\lambda(x) = a_N^{\tilde{\lambda}}(x) e^{-z_0 x N} \prod_{i=1}^N \left( M(\tilde{\lambda}_N^i) / M(\lambda_N^i) \right).$$

From the last equation we obtain

$$\begin{aligned} \frac{1}{N} \log a_N^\lambda \left( \frac{1}{N} \sum_{i \in \Lambda_N} \rho'(\tilde{\lambda}_N^i) \right) &= \frac{1}{N} \log a_N^{\tilde{\lambda}} \left( \frac{1}{N} \sum_{i \in \Lambda_N} \rho'(\tilde{\lambda}_N^i) \right) \\ &\quad - \frac{z_0}{N} \sum_{i \in \Lambda_N} \rho'(\tilde{\lambda}_N^i) + \frac{1}{N} \sum_{i \in \Lambda_N} \log \frac{M(\tilde{\lambda}_N^i)}{M(\lambda_N^i)}. \end{aligned} \quad (63)$$

By our choice of  $z_0(x)$  we find

$$\begin{aligned} \frac{1}{N} \sum_{i \in \Lambda_N} \rho'(\tilde{\lambda}_N^i) &= \frac{1}{N} \sum_{i \in \Lambda_N} \frac{M'(\lambda_N^i + z_0)}{M(\lambda_N^i + z_0)} \longrightarrow \\ &\int_0^1 \frac{M'(\lambda(\theta) + z_0)}{M(\lambda(\theta) + z_0)} d\theta = x \text{ as } N \rightarrow \infty. \end{aligned} \quad (64)$$

The statement of the theorem follows now from (62), (63), (64), and Lemma A.1 below.

**Lemma A.1.** *Let*

$$I_N^\lambda(y) = -\frac{1}{N} \log a_N^\lambda(y)$$

where  $a_N^\lambda$  defined as in Theorem A.2. Then for any  $y_1, y_2 \in \{|y| \leq R\}$

$$|I_N^\lambda(y_1) - I_N^\lambda(y_2)| \leq C(R, \|\lambda\|)|y_1 - y_2|.$$

**Proof.** We have

$$I_N^\lambda(y) = -\frac{1}{N} \log \int_{\mathbb{X}_{N,y}} \prod_{i=1}^N \frac{\exp(\lambda_N^i x_i - \varphi(x_i))}{M(\lambda_N^i)} d\sigma_{N,y} \quad (65)$$

Computing the derivative we find

$$\frac{d}{dy} I_N^\lambda(y) = -\frac{1}{N} \sum_{i \in \Lambda_N} \lambda_N^i + \frac{1}{N} \int_{\mathbb{X}_{N,y}} \sum_{i \in \Lambda_N} \varphi'(x_i) d\mu_{N,y}^\lambda(x). \quad (66)$$

where the measure  $d\mu_{N,y}^\lambda(x)$  is given by

$$\frac{d\mu_{N,y}^\lambda(x)}{d\sigma_{N,y}} = \frac{\exp\left(\sum_{i \in \Lambda_N} (\lambda_N^i x_i - \varphi(x_i))\right)}{\int_{\mathbb{X}_{N,y}} \exp\left(\sum_{i \in \Lambda_N} (\lambda_N^i x_i - \varphi(x_i))\right) d\sigma_{N,y}}.$$

The first term of (66) is bounded by  $\|\lambda\|$ . To estimate the second term we apply Jensen's inequality: for any  $\delta > 0$

$$\begin{aligned} \int_{\mathbb{X}_{N,y}} \sum_{i \in \Lambda_N} |\varphi'(x_i)| d\mu_{N,y}^\lambda(x) &\leq \frac{1}{\delta N} \log \int_{\mathbb{X}_{N,y}} e^{\delta \sum_{i \in \Lambda_N} |\varphi'(x_i)|} d\mu_{N,y}^\lambda(x) \\ &\leq \frac{1}{\delta N} \log \frac{\int_{\mathbb{X}_{N,y}} \exp\left(\sum_{i \in \Lambda_N} (\delta |\varphi'(x_i)| + \|\lambda\| |x_i| - \varphi(x_i))\right) d\sigma_{N,y}}{\int_{\mathbb{X}_{N,y}} \exp\left(-\sum_{i \in \Lambda_N} (\|\lambda\| |x_i| + \varphi(x_i))\right) d\sigma_{N,y}} \end{aligned}$$

Let

$$\begin{aligned} \varphi_{\delta,\lambda}(x) &= -\delta |\varphi'(x)| - \|\lambda\| |x| + \varphi(x) + \log C_{\delta,\lambda} \\ \varphi_\lambda(x) &= \|\lambda\| |x| + \varphi(x) + \log C_\lambda, \end{aligned}$$

where the constants  $C_{\delta,\lambda}$  and  $C_\lambda$  are chosen to turn  $\exp(-\varphi_{\delta,\lambda})$  and  $\exp(-\varphi_\lambda)$  into probability densities. Now we apply Theorem A.1 twice with  $\varphi$  replaced by  $\varphi_{\delta,\lambda}$  and  $\varphi_\lambda$  to see that

$$\lim_{N \rightarrow \infty} \frac{1}{\delta N} \log \int_{\mathbb{X}_{N,y}} e^{\delta \sum_{i \in \Lambda_N} |\varphi'(x_i)|} d\mu_{N,y}^\lambda(x) = \frac{1}{\delta} \log \frac{C_{\delta,\lambda}}{C_\lambda} + \frac{1}{\delta} (h_\lambda(y) - h_{\delta,\lambda}(y))$$

uniformly on compact  $y$ -intervals. Rate functions  $h_{\delta,\lambda}$  and  $h_\lambda$  are defined as in (17) with a suitable choice of index for  $\varphi$ . They are bounded on bounded  $y$ -intervals. This concludes the proof.  $\square$

## B One lemma from convex analysis.

**Lemma B.1.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^n$  and  $f$  and  $g$  be non-negative  $\mu$ -measurable functions. Assume also that  $\int g d\mu = 1$ . Then for any  $q \in [1, 2]$*

$$\int f^q g d\mu \leq C_q + 2^{2q-2} \log^q \left( \int e^f d\mu + 1 \right) + 2^{q-1} \int |\log g|^q g d\mu,$$

where  $C_q$  is a positive constant.

**Proof.** The statement of the lemma is a consequence of the convexity of the function

$$F(u) = e^{u^\alpha} - u^\alpha \quad \text{for } u \geq 0, \alpha = 1/q \in [1/2, 1]. \quad (67)$$

We write

$$\begin{aligned} \int f^q g d\mu &= \int_{f \geq \log g} f^q g d\mu + \int_{f < \log g} f^q g d\mu \leq \\ &\int_{f \geq \log g} f^q g d\mu + \int_{f < \log g} |\log g|^q g d\mu. \end{aligned} \quad (68)$$

Then we estimate the first term in the right-hand side of the above inequality.

$$\int_{f \geq \log g} f^q g d\mu \leq 2^{q-1} \int (f - \log g)_+^q g d\mu + 2^{q-1} \int_{f \geq \log g} |\log g|^q g d\mu. \quad (69)$$

By Jensen's inequality for function (67) we have

$$\exp\left(\int (f - \log g)_+^q g d\mu\right)^{1/q} - \left(\int (f - \log g)_+^q g d\mu\right)^{1/q} \leq \int (\exp(f - \log g)_+ - (f - \log g)_+) g d\mu. \quad (70)$$

Since  $u \leq (q/e)e^{u/q}$  for any  $q > 0$  and  $u \geq 0$ , we obtain from (70)

$$\begin{aligned} \exp\left(\int (f - \log g)_+^q g d\mu\right)^{1/q} &\leq \\ &\leq \frac{q}{e} \left(\int e^{(f - \log g)_+} g d\mu\right)^{1/q} + \int e^{(f - \log g)_+} g d\mu \\ &= \frac{q}{e} \left(\int_{f \geq \log g} e^f d\mu + \int_{f < \log g} g d\mu\right)^{1/q} + \int_{f < \log g} e^f d\mu + \int_{f < \log g} g d\mu \\ &\leq \frac{q}{e} \left(\int e^f d\mu + 1\right)^{1/q} + \int e^f d\mu + 1. \end{aligned} \quad (71)$$

Applying the function  $\log^q(\cdot)$  to both sides of (71) and taking into account the inequality  $cu + u^q \leq (c + 1)u^q$  for  $u \geq 1$ , we conclude that

$$\int (f - \log g)_+^q g d\mu \leq \log^q\left(\left(\frac{q}{e} + 1\right)\left(\int e^f d\mu + 1\right)\right). \quad (72)$$

Relations (68), (69) and (72) imply the statement of the lemma.  $\square$

## Acknowledgements

I am very grateful to Prof. S. R. S. Varadhan for many valuable discussions and suggestions. I would like also to thank Courant Institute of Mathematical Sciences for support.

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