

Insurance Equilibrium with Monoline and Multiline Insurers ^{*}

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Abstract

We study a competitive multiline insurance industry, in which insurance companies with limited liability choose which insurance lines to cover and the amount of capital to hold. The results are developed under the realistic assumptions that insurers face friction costs in holding capital and that the losses created by insurer default are shared among policyholders following an ex post, pro rata, sharing rule. We characterize the situations in which monoline and multiline insurance structures will be optimal. Markets characterized by a large number of essentially independent risks will be served by multiline firms. Markets for which the risks are asymmetric or correlated may best served by monoline insurers. The results are directly relevant to such catastrophe lines as bond and mortgage default insurance, and may be applicable more generally to industries in which risky activities can be carried out by either monoline or conglomerate entities. We illustrate the results with examples.

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1 Background

If insurers hold sufficient capital, they can make a credible guarantee to pay all claims. In practice, two factors create a significant risk of insurer default: 1) limited liability, creating conditions under which insurers may fail to make payments to policyholders and 2) excess costs to holding capital, leading insurers to limit the amount of capital they hold. When markets are incomplete, in the sense that policyholders face a counterparty risk that cannot be independently hedged, the existence of the two factors can have a significant impact on the industry equilibrium.

For example, Froot, Scharfstein, and Stein (1993) and Froot and Stein (1998) emphasize the importance of capital market imperfections for understanding a variety of corporate risk management decisions, with the tax disadvantages to holding capital within a firm an especially common and important factor. For insurance firms, Cummins (1993), Merton and Perold (1993), Jaffee and Russell (1997), Myers and Read (2001), and Froot (2007) all emphasize the importance of various accounting, agency, informational, regulatory, and tax factors in raising the cost of internally held capital. Such frictions combined with limited liability can have a significant impact on the amount of capital held by insurers, the premiums set across insurance lines, and the industry structure regarding which insurance lines are provided by monoline versus multiline insurers.

The risk of insurer default in paying policyholder claims has led to the imposition of strong regulatory constraints on the insurance industry in most countries. Capital requirements are one common form of regulation, although no systematic framework is available for determining the appropriate levels. As Cummins (1993) and Myers and Read (2001) point out, it is likely that the capital requirements are being set too high in some jurisdictions and too low in others, and similarly for the various lines of insurance risk, in both cases leading to inefficiency. It is thus important to have an objective framework for identifying the appropriate level of capital based on each

insurer's particular book of business.

Insurance regulation also focuses on the industry structure, requiring that certain high-risk insurance lines be provided on a monoline basis. A monoline restriction requires that the insurer dedicates its capital to pay claims on its single line of business alone, thus eliminating the diversification benefit in which a multiline firm can apply its capital to pay claims on any and all of its insurance lines.¹ Jaffee (2006), for example, describes the monoline restrictions imposed on mortgage default insurance, an industry, as it happens, currently at significant risk to default on its obligations as a result of the subprime mortgage crisis. Jaffee conjectures that the monoline restrictions were imposed as consumer legislation to protect the policyholders on relatively safe lines from an insurer default that would be created from large losses on catastrophe lines, such as mortgage and bond insurance. It is thus valuable to have a framework in which the optimality of monoline versus multiline formats can be determined.

This paper provides a systematic analysis of the structure of an insurance market with multiple lines of risk under the assumptions of costly capital, limited liability, incomplete markets and perfect competition among insurance companies. We specifically focus on the following questions:

1. How will risk-averse agents rank risks and evaluate monoline versus multiline coverage when insurance is available?
2. For a given choice of insurance lines, what is the equilibrium level of capital for an insurer to hold, and what are the resulting premiums net of default costs? We use the term "default costs" to refer to the shortfall in payments received by policyholders on claims. We do not assume any deadweight bankruptcy costs that might also be associated with such shortfalls.
3. How will insurers choose which insurance lines to offer to their customers, includ-

¹Monoline restrictions do not preclude an insurance holding company from owning an amalgam of both monoline and multiline subsidiaries. Within a holding company, the force of a monoline restriction is to restrict the capital of each monoline division to paying claims for that division alone.

ing the choice whether to operate on a monoline or multiline basis?

We develop a parsimonious model, which we use to analyze these questions. Our results significantly extend and generalize the analyses in earlier papers, e.g., in Phillips, Cummins, and Allen (1998) (PCA) and Myers and Read (2001) (MR) based on three key factors:

1. We consider a competitive market, in which insurance companies (insurers) compete to attract risk averse agents (insurees) who wish to insure risks. This competition severely restricts the monoline and multiline structures that may exist in equilibrium. We know of no other paper that provides an analytic framework for determining the equilibrium structure for an insurance industry that may contain both monoline and multiline firms.
2. We assume that there are excess costs associated with internal capital held by the insurer. PCA do model the risk of insurer default based on limited capital, but their analysis does not embed a cost of internal capital or other motivation for the limited capital. As a result, equilibrium premiums in their model have no component that reflects the internal cost of capital. MR apply the PCA premium model, so the same comment applies to their paper.
3. When insurer bankruptcy occurs, the PCA and MR papers assume that the asset shortfall is allocated to policyholders on the basis of the ex ante value of the default free insurance. While an ex ante rule provides analytic simplifications, it requires an impractical pattern of ex post side payments, for example that policyholders with no claims must make payments to other policyholders, see Ibragimov, Jaffee, and Walden (2008b). In contrast, in this paper we apply an ex post pro rata rule under which an asset shortfall is allocated in proportion to each policyholders actual claim. Ibragimov, Jaffee, and Walden (2008b) show that this ex post rule has sensible properties and reflects the actual industry practice.

Although our model is developed in the context of an insurance market, we believe the framework may be applicable to the issues of counterparty risk and monoline structures that are pervasive throughout the financial services industry. For example, the 1933 Glass Steagall Act forced U.S. commercial banks to divest their investment bank divisions, a clear monoline restriction. Subsequent legislation—specifically the 1956 Bank Holding Company Act and the Gramm-Leach-Bliley Act of 1999—provided more flexibility for bank holding companies, but the capital of the underlying commercial bank still may be used only to offset losses of that bank alone. In this sense, U.S. commercial banks remain monoline entities. In a similar fashion, Leland (2007) develops a model in which single-activity corporations can choose the optimal debt to equity ratio, whereas multiline conglomerates obtain a diversification benefit but can only choose an average debt to equity ratio for the overall firm. Thus here too there is a tension between the diversification benefit associated with a multiline structure and the benefit of separating risks allowed by a monoline structure.

The paper is organized as follows: In section 2, we introduce the basic framework for our analysis. In section 3, we analyze the monoline versus multiline choice and the implications for industry structure. We analyze the insurance line choices in a competitive market under two different assumptions for the properties of the underlying risks. In the first case, we assume that many essentially independent risks are available: insurance companies will then be multiline oriented. In the second case, we assume that the losses between lines are highly correlated, or that the loss distributions between lines are asymmetric, or that there are few lines overall: the market may then be best served by monoline insurance companies. We also provide several other results, e.g., a detailed analysis of capital choice in the single line case, as well as an extension of the second order stochastic dominance concept to the case when insurance markets are present. Finally, section 4 provides concluding remarks.

2 A competitive multiline insurance market

2.1 The special case with one insured risk class

We first study the case of only one insured risk class to introduce the basic concepts and notation.² Consider the following one-period model of a competitive insurance market. At $t = 0$, an *insurer* (i.e., an insurance company) in a competitive insurance market sells insurance against an idiosyncratic risk, $\tilde{L} \geq 0$ (throughout the paper we use the convention that losses take on positive values) to an *insuree*.³

The expected loss of the risk is $\mu_L = E[\tilde{L}]$, $\mu_L < \infty$. For many types of individual and natural disaster risks, such as auto and earthquake insurance, etc., it seems reasonable to assume that risks are idiosyncratic, although, of course, there will be some mega-disasters and corporate risks for which it is not true.

The insurer has limited liability and reserves capital within the company, so that the assets A are available at $t = 1$, at which point losses are realized and the insurer satisfies all claims by paying \tilde{L} to the insuree, as long as $\tilde{L} \leq A$. But, if $\tilde{L} > A$, the insurer pays A and defaults on the additional amount that is due.⁴ Thus, the payment is

$$\text{Payment} = \min(\tilde{L}, A) = \tilde{L} - \max(\tilde{L} - A, 0) = \tilde{L} - \tilde{Q}(A),$$

where $\tilde{Q}(A) = \max(\tilde{L} - A, 0)$, i.e., $\tilde{Q}(A)$ is the payoff to the option the insurer has to default. When obvious, we suppress the A dependence, e.g., writing \tilde{Q} instead of $\tilde{Q}(A)$. The price for the insurance is P . Throughout the paper, the risk-free discount rate

²For a more extensive discussion of the basic properties of the model, see Ibragimov, Jaffee, and Walden (2008b).

³It is natural to think of each risk as an insurance line. This interpretation is motivated if risks are perfectly correlated within an insurance line. Similar results arise if risks within a line are i.i.d., although the analysis becomes more complex.

⁴For some lines of consumer insurance (e.g. auto and homeowner), there exist state guaranty funds through which the insurees of a defaulting insurer are supposed to be paid by the surviving firms for that line. In practice, delays and uncertainty in payments by state guaranty funds leave insurees still facing a significant cost when an insurer defaults; see Cummins (1988). More generally, our analysis applies to all the commercial insurance lines and catastrophe lines for which no state guaranty funds exist.

is normalized to zero. The results are qualitatively the same with a non-zero risk-free rate.

With unlimited liability and no friction costs, the price for \tilde{L} risk in the competitive market is

$$P_L = \mu_{\tilde{L}},$$

Similarly, the value of the option to default is

$$P_Q = E[\tilde{Q}(A)] = \mu_Q.$$

We assume that these prices are determined in a friction-free competitive market for risk.

We assume, however, that there are friction costs when holding capital within an insurer, including both taxation and liquidity costs. The cost of internal capital is δ per unit of capital. This means that to ensure that A is available at $t = 1$, $(1 + \delta) \times A$ needs to be reserved at $t = 0$. Since the market is competitive and the cost of internal capital is δA , the price charged for the insurance is

$$P = P_L - P_Q + \delta A = \mu_L - \mu_Q + \delta A. \tag{1}$$

The premium setting and capital allocations build on the no-arbitrage, option-based, technique, introduced to insurance models by Doherty and Garven (1986), then extended to multiline insurers by Phillips, Cummins, and Allen (PCA, 1998) and Myers and Read (MR, 2001), and further developed in Ibragimov, Jaffee, and Walden (2008b). The results developed here apply to both monoline and multiline insurers.

We assume, in line with practice, that premiums are paid upfront, and thus to ensure that A is available at $t = 1$, the additional amount of $A - P_L + P_Q$ needs to be contributed by the insurer. Through the remainder of the paper, we shall refer to A as the insurer's assets or capital, depending on the context, it being understood that the amount $P_L - P_Q + \delta A$ is paid by the insurees as the premium, and the amount

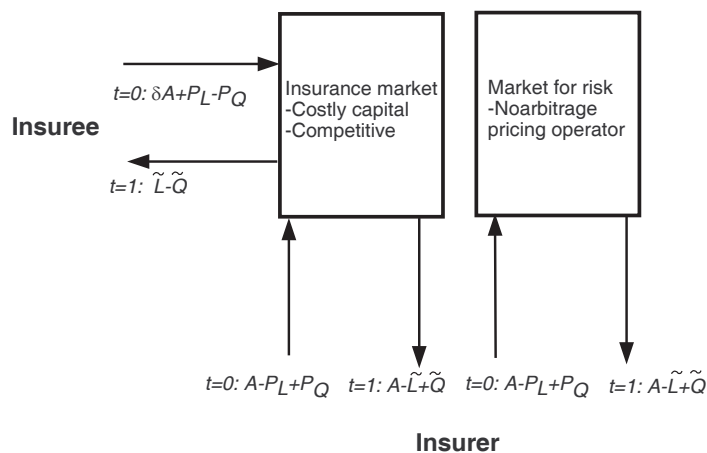


Figure 1: *Structure of model.* Insurers can invest in market for risk and in a competitive insurance market. There is costly capital, so to ensure that A is available at $t = 0$, $(1 + \delta)A$ needs to be reserved at $t = 1$. The premium, $\delta A + P_L - P_Q$, is contributed by the insuree and $A - P_L + P_Q$ by the insurer. The discount rate is normalized to zero. Competitive market conditions imply that the price for insurance is $P = P_L - P_Q + \delta A$.

$A - P_L + P_Q$ is contributed by the insurer's shareholders.⁵ The total industry structure is summarized in Figure 1.

It is natural to ask why insurees, recognizing that insurers impose the costs of holding internal capital, would not instead purchase their coverage directly in the market for risk. The answer is that here, as in any model of financial intermediation, there must be other costs, arising from transactions, contracting, or asymmetric information, which cause agents to prefer to deal with the intermediary. In this paper, we simply make the assumption that insurees do not have direct access to the market for risk and that they can obtain coverage only through the insurers.

⁵We do not explicitly consider the existence of reinsurance, which an insurer could use to transfer some of its risks. However, our results can be interpreted as applying to the insurer's risks net of those transferred to a reinsurer. Indeed, our analysis is equally applicable to insurers and reinsurers.

2.2 The general case of multiple insured risk classes

The generalization to the case when there are multiple risk classes requires an additional assumption regarding the timing at which claims are made. Here we follow PCA by assuming that claims on all the insured lines are realized at the same time, $t = 1$. The result is that at $t = 1$ the insurer either pays all claims in full (when assets exceed total claims) or defaults (when total claims exceed the assets). This is also the basis for our ex post sharing rule by which the shortfall in total assets for a defaulting insurer is allocated across insurance lines in proportion to the actual claims by line.

If coverage against N risks is provided by one multiline insurer, the total payment made to all policyholders with claims, taking into account that the insurer may default, is

$$\text{Total Payment} = \tilde{L} - \max(\tilde{L} - A, 0) = \tilde{L} - \tilde{Q}(A),$$

where $\tilde{L} = \sum_i \tilde{L}_i$ and $\tilde{Q}(A) = \max(\tilde{L} - A, 0)$. In the case of default under the ex post sharing rule, the payments made to insuree i is then

$$\text{Payment}_i = \frac{\tilde{L}_i}{\tilde{L}} A = \tilde{L}_i - \frac{\tilde{L}_i}{\tilde{L}} \tilde{Q}(A).$$

We define the binary default option

$$\tilde{V}(A) = \begin{cases} 0 & \tilde{L} \leq A, \\ 1 & \tilde{L} > A, \end{cases} \quad (2)$$

and the price for such an option in the competitive friction free market, $P_V = E[\tilde{V}]$. The total price for the risks is, $P \stackrel{\text{def}}{=} \sum_i P_i$, where P_i is the price for insurance against risk i . From PCA (1998) and MR (2001), and as extensively discussed in Ibragimov, Jaffee, and Walden (2008b), it follows that

$$P_i = P_{L_i} - r_i P_Q + v_i \delta A, \quad (3)$$

where

$$r_i = E \left[\frac{\tilde{L}_i}{\tilde{L}} \times \frac{\tilde{Q}}{P_Q} \right], \quad v_i = E \left[\frac{\tilde{L}_i}{\tilde{L}} \times \frac{\tilde{V}}{P_V} \right].$$

Now consider an insurance market, in which M insurers sell insurance against $N \geq M$ risks. We partition the total set of N risks into $\mathcal{X} = \{X_1, X_2, \dots, X_M\}$, where $\cup_i X_i = \{1, \dots, N\}$, $X_i \cap X_j = \emptyset, i \neq j$, $X_i \neq \emptyset$. The partition represents how the risks are insured by M monoline or multiline insurers.

The total industry structure is then characterized by the duple, $\mathcal{S} = (\mathcal{X}, \mathbf{A})$, where $\mathbf{A} \in \mathbb{R}_+^M$ is a vector with i :th element representing the capital available in the firm that insures the risks for agents in X_i . Here, we use the notation $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_- = \{x \in \mathbb{R} : x \leq 0\}$. We call \mathbf{A} the capital allocation and \mathcal{X} the industry partition. The number of sets in the industry partition is denoted by $M(\mathcal{X})$. Two polar cases are the fully multiline industry partition, $\mathcal{X}^{MULTI} = \{\{0, 1, \dots, N\}\}$ and the monoline industry partition, $\mathcal{X}^{MONO} = \{\{0\}, \{1\}, \dots, \{N\}\}$. Of course, for a fully multiline industry structure, $M = 1$ and $\mathbf{A} = A$. Given an industry structure, \mathcal{S} , the price in each line is uniquely defined through (3), $P_i = P_i(\mathcal{S})$.

To model the prevailing industry structure, $\mathcal{S} = (\mathcal{X}, \mathbf{A})$ —our main objective—we also need assumptions about insurees. For simplicity, we assume that there are N distinct insurees.⁶ Each risk is insured by one insuree with expected utility function u , where u is a strictly concave, increasing, twice continuously differentiable function defined on the whole of \mathbb{R}_- , and that $u'(0) \geq C > 0$, and $u''(x) \leq C < 0$, for some constant C , for all $x \leq 0$. For some of the results we need to impose stronger conditions on u . The risk can not be divided between multiple insurers.⁷ Finally, we assume that

⁶As mentioned in section 2, we do not distinguish between lines of risks and individual risks, in effect assuming that there is one insuree within each line. So far this is no restriction, since, in principle, N can be very large. If we wish to study a case with a “small” N , for the special case when there are several identical agents with perfectly correlated risks, we can treat such a situation as there being one representative insuree facing one risk, collapsing many risks into one line.

⁷Sharing risks is uncommon in practice, reflecting the fixed costs of evaluating risks and selling policies, as well as the agency problems between insurers when handling split insurance claims.

expected utility, U , is finite, $U = Eu_i(-\tilde{L}_i) > -\infty$ for all i .

We will make extensive use of the certainty equivalent as the measure of the size of a risk. For a specific utility function, u , the certainty equivalent of risk \tilde{L} , $CE_u(-\tilde{L}) \in \mathbb{R}_-$ is defined such that $u(CE_u(-\tilde{L})) = E[u(-\tilde{L})]$.

To summarize, the following assumptions are made about insurees and risks:

1. *Idiosyncratic risks*: The insurance risks are idiosyncratic.
2. *Limited liability*: Insurers have limited liability.
3. *Costly capital*: There is a cost for insurers to hold capital.
4. *Competitive insurance markets*: Prices for insurance are set competitively.
5. *Risk-averse insurees*: Insurees are risk averse.
6. *Nondivisibility*: Risks are nondivisible.

3 Industry structure

We now turn to our main objectives, namely to study how the industry structure—monoline versus multiline—and the related capital allocations are determined. To analyze these questions given a fixed level of capital and prices, although quite straightforward, may give misleading results because the capital held and the structure chosen are determined simultaneously. For example, an insurance company choosing to be massively multiline may choose to hold a lower level of capital than the total capital of a set of monoline firms insuring the same risks. A similar point for corporations with tax shields is made in Leland (2007), when analyzing the financial synergies of a merger between two firms, although the driving forces behind the results in Leland (2007) are, of course, different.

The differing risk structures between monoline and multiline insurers will also have implications for the pattern of cash flow payments received by the insurees. In a

competitive market we would expect differences in payments, due to the risk structure, to have pricing implications, since insurees will value each cash flow pattern differently. Therefore, before answering the questions of industry structure, we study how total capital, A , and price, P , are endogenously determined in a competitive market.

In section 3.1, we start by analyzing these questions for competitive monoline insurers with costly internal capital, $\delta > 0$. In section 3.2 we focus on the concept of risk rankings when insurance is present (question 1 in the introduction). Then in sections 3.3 and 3.4, we extend the analysis of section 3.1 to the multiline setting, which allows us to analyze the capital choice (question 2) and industry structure (question 3).

3.1 Capital and price in the monoline case

We analyze what price will be charged for insurance, as a function of the level of capital, by an insurance company offering insurance in a single insurance line in a competitive market. We then use the pricing analysis to understand what level of capital will be chosen for such an insurance company.

For simplicity, in this section we assume that the loss, \tilde{L} , is absolutely continuous, with a strictly positive probability density function (p.d.f.) on the whole of \mathbb{R}_+ . We define the default option's "Eta", $\eta(A) = \frac{\partial E[\tilde{Q}(A)]}{\partial A}$. Since \tilde{L} has an absolutely continuous, strictly positive, distribution, $\eta(A)$ is a continuous strictly negative function on $(0, \infty)$ regardless of the distribution of \tilde{L} (Ingersoll 1987). We study the price of insurance, P as a function of capital, A . It is straightforward to show that

Lemma 1 *If \tilde{L} has an absolutely continuous distribution, with support on the whole of \mathbb{R}_+ , then the price of insurance as a function of capital, A , satisfies the following conditions*

1. $P(0) = 0$,
2. $P'(A) = \delta - \eta(A) > 0$,

3. $P(A) = \mu_L + \delta A + o(1)$, for large A ,

4. $P''(A) < 0$.

Here, for a general function, $f(A) = o(1)$ means that $\lim_{A \rightarrow \infty} f(A) = 0$. Thus, regardless of the distribution of \tilde{L} , $P(A)$ will be a strictly increasing, strictly concave function with known asymptotics, as shown in Figure 2.

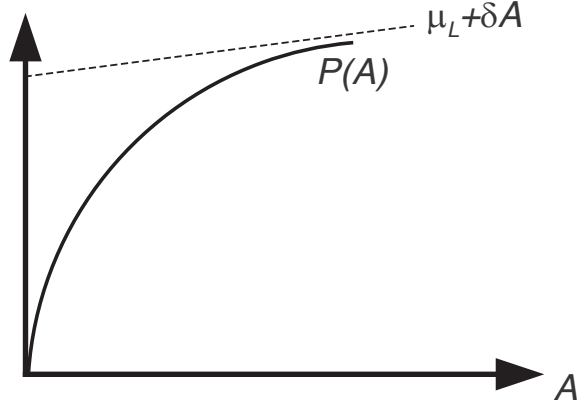


Figure 2: Insurance premium, P_L as a function of capital, A , for an arbitrary absolutely continuous risk, \tilde{L} , with support on $(0, \infty)$. The line $\mu_L + \delta A$ is the price of insurance without default (i.e with unlimited liability). The curve $P(A)$ is the premium net of default costs. The distance between the two lines is thus the value of the option to default.

The conditions in Lemma 1 are natural. The first condition states that if the insurer does not put aside any capital, it may charge no premium (anything else would be an arbitrage opportunity). The second condition shows that for a small increase in capital, A the premium, P , increases via two effects: the cost of internal capital, δA , increases, and the value of the default option decreases ($\eta < 0$). The third condition shows that as A becomes large, the premium approaches the sum of the friction free price of insurance with unlimited liability, $P_L = \mu_L$, (since the option value of defaulting disappears) and the cost of holding internal capital. The second term becomes large, since it is proportional to capital. The fourth condition, which follows as a direct

consequence of the convexity of an option's value as a function of strike price (see Ingersoll 1987), states that P is concave.

The optimal (A, P) pair will depend on the preferences of the insuree. We therefore turn to the insuree's problem. The price the insuree has to pay, from (1), is

$$P(A) = \mu_L - \mu_Q + \delta A.$$

Given the competitiveness in the insurance market, the insurer will choose capital, A that maximizes the expected utility of the insuree, i.e., since the total payoff to the insuree is $-P(A) - \tilde{L} + (\tilde{L} - \tilde{Q}(A)) = -P(A) - \tilde{Q}(A)$,

$$A^* = \arg \max_{0 \leq A < \infty} Eu[-P(A) - \tilde{Q}(A)]. \quad (4)$$

If an insurer were to select a value for capital, A other than A^* , this would allow a competitor to outcompete the insurer by offering a contract with a preferable level of capital, A^* .

In general, A^* may be a set, i.e., there can be multiple solutions to (4). If $\delta = 0$, it is easy to show that the company will reserve an arbitrary large amount of capital. Formally, the solution is $A^* = \{\infty\}$ and the price is $P = \mu_L$. We call this the *friction-free outcome*, since the insurer never defaults and all risk is transferred from the insuree to the insurer in an optimal manner. In this case the expected utility of the insuree is $U = u(-\mu_L)$ and the certainty equivalent of his decrease in utility is the same as if he were risk-neutral, $CE_u(\tilde{L}) = -\mu_L$, since μ_L is exactly the premium he pays for full insurance.

When capital is costly, $\delta > 0$, it is not possible to obtain the friction-free outcome. We assume that the cost of holding capital is small compared with expected losses. Specifically, we assume that

Condition 1 $CE_u(-P(A) - \tilde{Q}(A)) < -\mu_L(1 + \delta)$ for all $A \in [0, \mu_L]$.

This condition implies that each risk is potentially insurable in that if an insurer could guarantee default-free insurance against a risk by holding capital just equal to the expected loss and by setting a premium equal to the expected loss plus the cost of holding the internal capital, the insuree would purchase such insurance. No other policy could be as attractive to the insuree. Generally, an insurer holding capital just equal to the expected loss would face a risk of default, and in practice insurers would hold a higher level of capital and there would still be a risk of default. Thus, given costly capital, the best possible risk-free outcome is for the insuree to reach a certainty equivalent of $-\mu_L(1 + \delta)$. We therefore call an outcome in which an insuree obtains $CE_u = -\mu_L(1 + \delta)$ the *ideal risk-free outcome with costly internal capital*.

It is easy to show that the set of solutions to (4) is compact and nonempty. However, it may be that $0 \in A^*$, i.e., that it is optimal not to be insured. In fact, for insurees that are close to risk neutral, we would expect insurance to be suboptimal, since the costs of internal capital would always be greater than the gain from the reduced risk. We wish to understand in which situations there is a potential for insurance to exist, i.e., when there exists a utility function such that $0 \notin A^*$. We have

Proposition 1

For a risk \tilde{L} and cost of holding internal capital, $\delta > 0$, there exists a strictly concave utility function, u , such that $0 \notin A^$ for an insuree with utility function u , if and only if there is a level of capital, A , such that the price, P , defined in (1), satisfies $P < A$.*

The “only if”-part of the proposition is immediate, since if it does not hold it would be less expensive for the insuree to hold the capital than to buy the insurance. Clearly, we would only expect such “self insurance” to be optimal when δ is large. The “if”-part is proved in the appendix.

There are several other observations that are straightforward to show. First, if \tilde{L} has an absolutely continuous distribution in a neighborhood of 0, then the optimization

in (4) satisfies $\partial Eu[-P(A) - \tilde{Q}(A)]/\partial A < 0$ at $A = 0$, i.e., insurees are always strictly worse off buying a small amount of insurance than buying no insurance at all. Second, if \tilde{L} has a bounded range, with upper bound \bar{L} , and \tilde{L} has an absolutely continuous distribution function in a neighborhood of \bar{L} , then $\partial Eu[-P(A) - \tilde{Q}(A)]/\partial A < 0$ at $A = \bar{L}$, i.e., insurees are always strictly worse off buying full insurance compared with buying slightly less than full insurance. These results are similar to the classical results in the insurance literature on optimal insurance contracts having deductibles. Third, if the p.d.f. of \tilde{L} vanishes on an interval $[a, b]$, then $Eu[-P(A) - \tilde{Q}(A)]$ is a strictly concave function of A for $A \in [a, b]$.

Thus, in the simplest case of a risk that has a scaled Bernoulli distribution, i.e., that takes on value $Z > 0$ with probability p and 0 with probability $1 - p$, $\tilde{L} \in Be(Z, p)$, the optimization problem (4) is concave, which allows for a complete characterization of the solution. We have

Proposition 2 *Let $\tilde{L} \sim Be(Z, p)$ and $\tilde{L}' \sim Be(Z', p)$, $Z' > Z$. Further, let A^* and $A^{*'}$ be the optimal internal capital for the risks \tilde{L} and \tilde{L}' , respectively. Then*

1. *The optimal internal capital is unique, and satisfies $A^* \in [0, Z]$.*
2. *If $A^{*'} = 0$ then $A^* = 0$.*
3. *If $A^* > 0$ then $A^{*'} > A^*$.*
4. *For an insuree with decreasing absolute risk aversion (DARA), $P_Q(A^*) < P_Q(A^{*'})$.*
5. *For an insuree with increasing absolute risk aversion (IARA), $P_Q(A^*) > P_Q(A^{*'})$.*

In case of general risks, none of 1-5 in proposition 2 need to hold, the technical reason being that the optimization problem may no longer be concave. This is true, even for the only slightly more complicated case, in which \tilde{L} has a three point distribution.

3.2 Ranking of risks when insurance markets are present

Our first objective is to understand, given that the insurance market is present, whether it is possible to rank risks, in the sense that any risk averse insuree agrees which risk is the worst of two risks. Of course, without an insurance market, stochastic dominance arguments can be used: Given two risks, with payoff $-\tilde{L}_1$ and $-\tilde{L}_2$, with $\mu_{L_1} = \mu_{L_2} = \mu_L$, $Eu(-\tilde{L}_1) \geq Eu(-\tilde{L}_2)$ for all utility functions, if and only if $-\tilde{L}_1$ second order stochastically dominates $-\tilde{L}_2$,

$$-\tilde{L}_1 \succeq -\tilde{L}_2. \quad (5)$$

If F_1 and F_2 are the c.d.f.'s of $-\tilde{L}_1$ and $-\tilde{L}_2$ respectively (with range in \mathbb{R}_-), it was shown in Rothschild and Stiglitz (1970) that $-\tilde{L}_1 \succeq -\tilde{L}_2$ is equivalent to the integration condition:

$$\int_{-\infty}^t F_1(x)dx \leq \int_{-\infty}^t F_2(x)dx,$$

for all $t < 0$.

Is there a similar ranking when the insurance market is present? To analyze this question, we define \tilde{Q}_1 and \tilde{Q}_2 as the option payoffs from default, for risk 1 and 2 respectively. In what follows, we restrict our attention to cases in which it is optimal for an insurer to buy insurance against risk \tilde{L}_2 and the optimal internal capital is greater than the expected loss, i.e., $A^* > \mu_L$. This is obviously a situation that we expect to have in a standard insurance setting.

We use (1) to get

$$U = Eu[-P - \tilde{Q}] = Eu[-\mu_L - \delta A + (\mu_Q - \tilde{Q})] \quad (6)$$

For a given A , (6) implies that regardless of utility function, an investor will be

better off facing risk \tilde{L}_1 , than \tilde{L}_2 if, for all $A > \mu_L$,

$$-(\tilde{Q}_1 - \mu_{Q_1}) \succeq -(\tilde{Q}_2 - \mu_{Q_2}). \quad (7)$$

Clearly, (7) is not the same as (5), so second order stochastic dominance does not immediately allow us to rank the risks in the presence of an insurance market. A stronger condition, however, is sufficient for such a ranking, as shown in the following

Proposition 3 *Given an insuror with a strictly concave utility function, in the presence of an insurance market. Consider two risks, \tilde{L}_1 and \tilde{L}_2 , with the same expected losses, $\mu_{L_1} = \mu_{L_2} = \mu_L$, such that the optimal capital for \tilde{L}_2 satisfies $A \in A_2^*$, $A > \mu_L$. Then, if for all $t < -\mu_L$,*

$$\int_{-\infty}^t F_1(x) dx \leq \int_{-\infty}^{t-\mu_L} F_2(x) dx, \quad (8)$$

the insuror is (weakly) better off facing risk \tilde{L}_1 than risk \tilde{L}_2 .

The following example shows the differences between what is required for second order stochastic dominance and the stronger condition that is needed for one insurable risk to dominate another.

Example: Consider the risks \tilde{L}_β , $\beta \geq 1$, where the c.d.f. of $-\tilde{L}_\beta$ is $F_\beta(x) = e^{\beta(x+1)-1}$, $x < 1/\beta - 1$ that are shifted, reflected, exponential distributions (the restriction $\beta \geq 1$ can be extended to $\beta > 0$ if \tilde{L} is allowed to take on negative values). It is clear that $\mu_L = E[\tilde{L}_\beta] = 1$ and, furthermore, it is easy to check that for $\beta_1 > \beta_2$, $F_{\beta_1}(x) - F_{\beta_2}(x-1) < 0$ for $x < -\beta_2/(\beta_1 - \beta_2)$, and $F_{\beta_1}(x) - F_{\beta_2}(x-1) \geq 0$ otherwise. Therefore $\int_{-\infty}^t (F_{\beta_1}(x) - F_{\beta_2}(x-1)) dx$ realizes a maximal value at $t = 1/\beta - 1$, and it is straightforward to check that

$$\beta_1 \geq \beta_2 e^{\beta_2}$$

is a necessary and sufficient condition for the conditions in Proposition 3 to be satisfied. This is obviously a stronger condition than $\beta_1 \geq \beta_2$, which is what is needed for second order stochastic dominance of the uninsurable risks, $-\tilde{L}_1 \succeq -\tilde{L}_2$.

3.3 The monoline versus multiline business choices

We now have almost all of the machinery to understand industry structure and capital choice, but we still need to extend the notion of a competitive market to a multiline setting.

We have already used the assumption of competitive markets to understand the pricing and optimal capital level in the monoline case. Specifically, equation (4) determined the level of capital held by competitive monoline insurers as the amount that maximizes insuree utility. In the multiline case, however, the analysis is more complex, since there are multiple possible industry structures, \mathcal{S} , and for each multiline structure a trade-off in utility levels must be evaluated for the participating insurees.

To impose additional restrictions, we first note that for N risks, $\tilde{L}_1, \dots, \tilde{L}_N$, and a general industry structure, $\mathcal{S} = (\mathcal{X}, \mathbf{A})$, when the ex post sharing rule is used, the residual risk for an insuree, $i \in X_j$, is

$$\tilde{K}_i(\mathcal{S}) = \frac{\tilde{L}_i}{\sum_{i' \in X_j} \tilde{L}_{i'}} \min \left(A_i - \sum_{i' \in X_j} \tilde{L}_{i'}, 0 \right). \quad (9)$$

His expected utility is therefore $E u_i(-P_i(\mathcal{S}) + \tilde{K}_i(\mathcal{S}))$.

For N agents with utility functions, u_i , $1 \leq i \leq N$, each agent wishing to insure risk \tilde{L}_i , an industry structure, \mathcal{S}' , Pareto dominates another industry structure, \mathcal{S} , if $E[u_i(-P_i(\mathcal{S}) + \tilde{K}_i(\mathcal{S}))] \leq E[u_i(-P_i(\mathcal{S}') + \tilde{K}_i(\mathcal{S}'))]$ for all i and $E[u_i(-P_i(\mathcal{S}) + \tilde{K}_i(\mathcal{S}))] < E[u_i(-P_i(\mathcal{S}') + \tilde{K}_i(\mathcal{S}'))]$ for at least one i . We also say that \mathcal{S}' is a Pareto improvement of \mathcal{S} . An industry structure, \mathcal{S} , for which there is no Pareto improvement is said to be Pareto efficient. An industry structure, $(\mathcal{X}, \mathbf{A})$, is said to be constrained Pareto efficient (given \mathcal{X}), if there is no \mathbf{A}' such that $(\mathcal{X}, \mathbf{A}')$ is a Pareto improvement of $(\mathcal{X}, \mathbf{A})$. An industry partition \mathcal{X} is said to be Pareto efficient, if there is a capital allocation, \mathbf{A} , such that $(\mathcal{X}, \mathbf{A})$ is Pareto efficient. In a Pareto dominated industry structure, we would expect insurers to enter the market with improved offerings, thereby outcompeting existing insurers. Such an outcome can therefore not be an equilibrium.

In fact, we impose a somewhat stronger requirement, restricting our attention to only those multiline outcomes for which a monoline entrant cannot increase the expected utility for any of the participating insurees (whether or not this makes some other agents worse off).

Definition 1

- *An industry structure, \mathcal{S} , is said to be robust to monoline blocking, if there is no insuree, $i \in \{1, \dots, N\}$ such that $E[u_i(-P_i(\mathcal{S}) + \tilde{K}_i(\mathcal{S}))] < E[u_i(-P(A) - \tilde{Q}(A))]$ for some $A \geq 0$, where $P(A)$ is the price of monoline insurance with internal capital A , and $\tilde{Q}(A)$ is the payout of the default option of such an insurance.*
- *The set of Pareto efficient industry structures robust to monoline blocking is denoted by \mathcal{O} .*
- *If, $(\mathcal{X}, \mathbf{A}) \in \mathcal{O}$, for some \mathbf{A} , we write $\mathcal{X} \in \mathcal{O}$.*

The concept of robustness to monoline blocking has similarities to the *core* concept used in coalition games (see, e.g., Osborne and Rubinstein (1984)), although, in general, \mathcal{O} is neither a subset, nor a superset of the core. In our model, monoline structures may dominate multilines, leading to non-cohesiveness, which means that the core may contain Pareto-dominated outcomes. Therefore, there may be outcomes in the core that are not in \mathcal{O} . On the other hand, elements in the core are robust to blocking/competition by any type of insurance company (monoline or multiline) which is a stricter requirement than the monoline blocking condition for \mathcal{O} , and moreover, \mathcal{O} may contain other structures than the partition into one massively multiline business, so \mathcal{O} may contain elements that are not in the core. In the case of cohesive games, the core is a subset of \mathcal{O} , since any element in the core will be Pareto efficient. If, in addition, there are only two insurance lines, the core is the same as \mathcal{O} , since only monoline blocking is possible in this case. It is easy to show that \mathcal{O} is always nonempty,

as opposed to the core. We are interested in \mathcal{O} , since we believe that it may be easier for a competitor to compete for customers within one line of business than in multiple lines simultaneously.

What can we say about industry structure when there are many risks available? Intuitively, when capital is costly and there are many risks available, we would expect an insurer to be able to diversify by pooling many risks and, through the law of large numbers, choose an efficient A^* per unit of risk. Therefore, the multiline structure should be more efficient than the monoline structure.⁸ The argument is very general, as long as there are enough risks to pool, and these risks are not too dependent. For example, in our model, under general conditions, the multiline business can reach an outcome arbitrary close to the ideal risk-free outcome with costly internal capital. We have:

Proposition 4 *Consider a sequence of insurees, $i = 1, 2, \dots$, with expected utility functions, $u_i \equiv u$, holding independent risks \tilde{L}_i . Suppose that u'' is bounded by a polynomial of degree q , $E\tilde{L}_i^p \leq C$ for $p = 2 + q + \epsilon$ and some $C, \epsilon > 0$, and $E(\tilde{L}_i) \geq C'$, for some $C' > 0$. Then, regardless of the cost of internal capital, $0 < \delta < 1$, as the number of risks in the economy, N , grows, a fully multiline industry, $\mathcal{X}^{MULTI} = \{\{1, \dots, N\}\}$ with capital $A = \sum_{i=1}^N \mu_{L_i}$, reaches an outcome that converges to the ideal risk-free outcome with costly internal capital, i.e.,*

$$\min_{1 \leq i \leq N} CE_u(-P_i((\mathcal{X}^{MULTI}, A)) + \tilde{K}_i((\mathcal{X}^{MULTI}, A))) = -\mu_{L_i}(1 + \delta) + o(1).$$

Proposition 4 can be generalized in several directions, e.g., to allow for dependence. As follows from the proof of the proposition, it also holds for all (possibly dependent) risks \tilde{L}_i with $E|\tilde{L}_i|^p < C$ that satisfy the Rosenthal inequality (see Rosenthal (1970)). The Rosenthal inequality and its analogues are satisfied for many classes of de-

⁸This type of diversification argument is, for example, underlying the analysis and results in both Jaffee (2006) and Lakdawalla and Zanjani (2006).

pendent random variables, including martingale-difference sequences (see Burkholder (1973) and de la Peña, Ibragimov, and Sharakhmetov (2003) and references therein), many weakly dependent models, including mixing processes (see the review in Nze and Doukhan (2004)), and negatively associated random variables (see Shao (2000) and Nze and Doukhan (2004)).

Using the Phillips-Solo device (see Phillips and Solo (1992)) in a similar fashion of the proof of Lemma 12.12 in Ibragimov and Phillips (2004), one can show that Proposition 4 also holds for correlated linear processes $\tilde{L}_i = \sum_{j=0}^{\infty} c_j \epsilon_{i-j}$, where (ϵ_t) is a sequence of i.i.d. random variables with zero mean and finite variance and c_j is a sequence of coefficients that satisfy general summability assumptions. Several works have focused on the analysis of limit theorems for sums of random variables that satisfy dependence assumptions that imply Rosenthal-type inequalities or similar bounds (see Serfling (1970), Móricz, Serfling, and Stout (1982) and references therein). Using general Burkholder-Rosenthal-type inequalities for nonlinear functions of sums of (possibly dependent) random variables (see de la Peña, Ibragimov, and Sharakhmetov (2003) and references therein), one can also obtain extensions of Proposition 4 to the case of losses that satisfy nonlinear moment assumptions.

Proposition 4 shows that, with enough risks, a solution can be obtained arbitrarily close to the ideal risk-free outcome with costly internal capital. Proposition 5 below shows the opposite, that with too few risks it is not possible to get arbitrarily close to the ideal risk-free outcome with costly internal capital:

Proposition 5 *Consider a sequence of insurances, $i = 1, 2, \dots$. If, in additions to the assumptions of proposition 4, the risks are uniformly bounded: $\tilde{L}_i \leq C_0 < \infty$ (a.s.) for all i , and $\text{Var}(\tilde{L}_i) \geq C_1$, for some $C_1 > 0$, for all i , then for every $\epsilon > 0$, there is an n such that $\lim_{\epsilon \searrow 0} n(\epsilon) = \infty$ and such that, as N grows, any partition with*

$\mathbf{A}_j = \sum_{i \in X_j} \mu_{L_i}$ for all j and

$$\min_{1 \leq i \leq N} CE_u(-P_i((\mathcal{X}, \mathbf{A})) + \tilde{K}_i((\mathcal{X}, \mathbf{A}))) \geq \mu_{L_i}(1 + \delta_i) - \epsilon, \quad (10)$$

must have $|X| \geq n$ for all $X \in \mathcal{X}$, i.e., any $X \in \mathcal{X}$ must contain at least n elements.

The condition of uniformly bounded risks in Proposition 5 can be relaxed. If the utility function, u , has decreasing absolute risk aversion, then the proposition holds if the *expectations* of the risks are uniformly bounded ($E[\tilde{L}_i] < C$) for all i .

The previous results show that it is possible to get close to the ideal risk-free outcome with costly internal capital, but only with massively multiline industry structures. The results indicate that such massively multiline industry structures provide diversification benefits for the insurees. However, since capital is costly, it may be that the insurees may not wish to reach the the ideal risk-free outcome with costly internal capital. They may be better off if a lower level of capital is chosen, facing some risk, but avoiding some cost of internal capital. Proposition 6 uses Pareto dominance arguments to show that, under additional assumptions about the risks, massively multiline industry structures Pareto dominate purely monoline outcomes, and that every industry structures with more than a few monoline insurers is Pareto dominated.

We need a precise definition of what it means for an industry to be massively multiline: An industry partition is said to be massively multiline if, as the number of lines in the industry, N , grows, the average number of lines per insurer grows without bounds, i.e., $\lim_{N \rightarrow \infty} N/M(\mathcal{X}) = \infty$. Here, $M(\mathcal{X})$, defined in section 2.2, is the number of insurers in the economy. We let \mathcal{X}^{MASS} (or \mathcal{X}_N^{MASS} , if we want to stress the number of risks) represent a massively multiline industry partition.

For some of the results, we need conditions on the behavior of the risks. Specifically, the following condition ensures that they are not too “asymmetric.” We impose the following conditions

Condition 2 *There are positive, functions, $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that, g is strictly decreasing, h is nonincreasing, $\lim_{x \rightarrow \infty} h(x) = 1$, and for all i ,*

$$F_i(x) \in [g(h(x)x), g(x)].$$

Here, $F_i(x) \stackrel{\text{def}}{=} \mathbb{P}(L_i \geq x)$.

We also need the risks to be insurable in the sense that the cost of capital is not too high and/or the insurees are too close to risk-neutral to wish to insure the risks. We could assume that the outcome in which no insurance is offered is not Pareto optimal, but we need a stronger condition

Condition 3 *There is an $\epsilon > 0$, such that, for each risk, i , the optimal capital for a monoline insurer is greater than ϵ , i.e., $a > \epsilon, \forall a \in A^*$, for all i .*

Both conditions ensure, in different ways, that the risks do not become degenerate for large i . Obviously, if the risks are i.i.d., $g = F_1$ and $h(x) \equiv 1$ can be chosen in condition 2.

We now have

Proposition 6 *Under the conditions of proposition 4, if the risks, \tilde{L}_i , have absolutely continuous distributions with strictly positive p.d.f.'s on \mathbb{R}_+ . Then, for large N ,*

1. *If conditions 2 and 3 are satisfied, there is a massively multiline industry partition, $\mathcal{X}^{\text{MASS}}$, that Pareto dominates the monoline industry partition, $\mathcal{X}^{\text{MONO}}$, i.e., $\mathcal{X}^{\text{MONO}} \notin \mathcal{O}$.*
2. *If conditions 2 and 3 are satisfied, there is a constant, $C < \infty$, such that any Pareto efficient industry partition has at most C monoline insurers, regardless of N , i.e., any $\mathcal{X} \in \mathcal{O}$ has at most C monoline insurers.*

3. *If the risks are i.i.d. and if condition 3 is satisfied, the fully multiline industry partition, \mathcal{X}^{MULTI} , Pareto dominates the monoline industry partition, \mathcal{X}^{MONO} .*
4. *If condition 1 is satisfied, the fully multiline industry partition, \mathcal{X}^{MULTI} with capital $A = \sum_i \mu_{L_i}$ Pareto dominates the monoline industry partition, \mathcal{X}^{MONO} .*

All these results are for economies in which many lines are present, with independent risks that are not too asymmetric. The first result shows that there are massively multiline industry structures that dominate monoline structures. The second result shows that a monoline structure be optimal only for very few lines, in a large economy. The third result shows that if the risks are identically distributed, then a market with one fully multiline insurer dominates the monoline outcome, and the fourth results show assumptions under which a firm with almost no risk dominates the monoline structure.

The difference between the first and third result in proposition 6 is important. If the risks are identically distributed, then the agents, having the same utility functions, will all agree upon the optimal level of internal capital, A^* . They may therefore agree to insure in one fully multiline company. If, on the other hand, the risks have different distributions (or equivalently, if the utility functions are different), then the insurees will typically disagree about what is the optimal level of internal capital. Recall that increasing capital has two offsetting effects. It decreases the risk of insurer default and thereby increases the expected utility of the insurees, but it also increases the total cost of internal capital, and thereby decreases the expected utility. For severe risks, as shown in a special case in section 3.1, it may be argued that a higher degree of internal capital will be optimal than for less severe risks. With different types of risks, it may therefore be optimal to have several massively multiline insurance firms that all choose different levels of internal capital, instead of one fully multiline company.

These asymptotic results together suggest that when there is a large number of essentially independent risks that are thin-tailed, a monoline insurance structure is

never optimal and that massively multiline industry structures may instead occur. For standard risks — like auto and life insurance — it can be argued that these conditions are reasonable. However, the results also provide an indication of when a multiline structure may not be optimal:

Implication 1 *A multiline industry structure may be suboptimal*

- *If the number of risks is limited.*
- *If risks are asymmetric, for example, when some risks are heavy-tailed and others are not.*
- *If risks are dependent.*

Catastrophe risks, in particular, appear to satisfy all these conditions under which a multiline structure may be suboptimal. Consider, for example, residential insurance against earthquake risk in California.⁹ The outcome for different households within this area will obviously be heavily dependent when an earthquake occurs, making the pool of risks essentially behave as one large risk, without diversification benefits. Moreover, many other catastrophic risks are known to have heavy tails. This further reduces the diversification benefits, even when risks are independent. Thus, even though an earthquake in California and a hurricane in Florida may be considered independent events, the gains from diversification of such risks may be limited due to their heavy-tailedness.

We now provide an example in which monoline insurance is more likely to dominate when the conditions of Implication 1 holds. Specifically, we show that asymmetry between risks and dependence of risks can lead to the monoline outcome being optimal.

⁹See, e.g., Ibragimov, Jaffee, and Walden (2008a).

3.4 One versus two lines - An example

We focus on the special and simplified case with two insurance lines and compare the two industry structures: $\mathcal{X}^{MONO} = \{\{1\}, \{2\}\}$ (monoline) versus $\mathcal{X}^{MULTI} = \{\{1, 2\}\}$ (multiline).

In the first partition, we know how $\mathbf{A}^1 = (A_1, A_2)^T$ should be chosen from our previous analysis, leading to industry structure $\mathcal{S}^{MONO} = (\mathcal{X}^{MONO}, \mathbf{A}^1)$. In the second partition, there is a whole range of capital, $A \in [\underline{A}, \overline{A}]$, leading to competitive outcomes, $\mathcal{S}^{MULTI} = (\mathcal{X}^{MULTI}, A)$.

The condition for the multiline structure to be optimal is now that there is an $A \in [\underline{A}, \overline{A}]$, such that \mathcal{S}^{MULTI} offers an improvement for both agents, i.e., $Eu[-P_i(\mathcal{S}^{MONO}) + \tilde{K}_i(\mathcal{S}^{MONO})] \leq Eu[-P_i(\mathcal{S}^{MULTI}) + \tilde{K}_i(\mathcal{S}^{MULTI})]$, $i = 1, 2$. We study the conditions under which this is satisfied.

For simplicity, we assume that insurees have expected utility functions defined by $u(x) = -(-x + t)^\beta$, $\beta > 1$, $x < 0$, and that \tilde{L}_1 and \tilde{L}_2 have Bernoulli distributions: $\mathbb{P}(\tilde{L}_1 = 1) = p$, $\mathbb{P}(\tilde{L}_2 = 1) = q$, $\text{corr}(\tilde{L}_1, \tilde{L}_2) = \rho$. Depending on $0 < p < 1$ and $0 < q < 1$, there are restrictions on the correlation, ρ . For example, ρ can only be equal to 1 if $p = q$.

We study the case $\beta = 7$, $t = 1$, $\delta = 0.2$ and $p = 0.25$. We first choose $q = 0.65$ and compare the monoline outcome with the multiline outcome for $\rho \in \{0.1, 0.2, 0.3, 0.4\}$, as shown in Figure 3. The solid vertical and horizontal lines show optimal expected utility for insuree 1 and 2 respectively when the industry is structured as two monoline firms (with optimal capital levels $A_1 = -12.06$ and $A_2 = -933.7$).

For the case of negative and zero correlation, the situation can be improved for both insuree classes by moving to a multiline (i.e., duo-line) solution, reaching an outcome somewhere on the efficiency frontier of the multiline utility possibility curve. For the case of $\rho = 0.1$, insuree class 1 will not participate in the multiline solution, and the monoline outcome will therefore prevail.

In Figure 4 we plot the regions in which the monoline and multiline solutions will

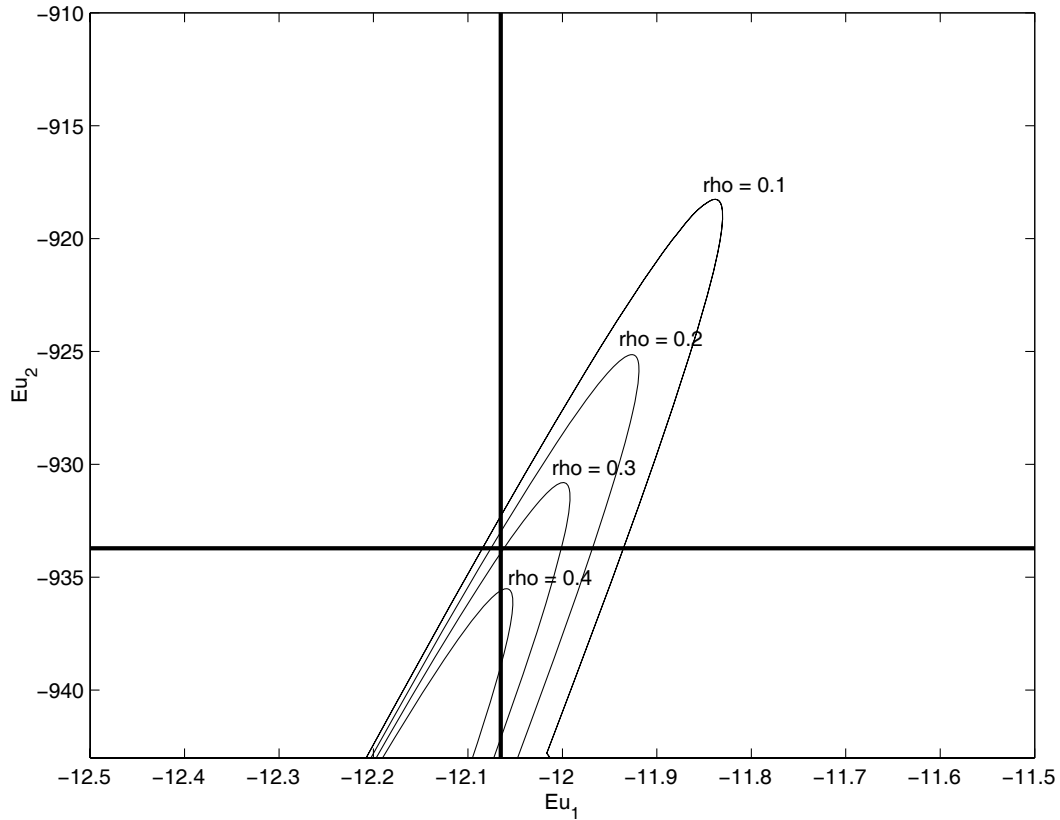


Figure 3: *The solid vertical and horizontal lines show optimal expected utility for insuree 1 and 2 respectively when the industry is structured as two monoline firms (with optimal capital levels $A_1 = -12.06$ and $A_2 = -933.7$). The curved lines show the utility combinations for a multiline insurer, based on 4 different correlations between risks 1 and 2. The monoline outcome dominates when $\rho = 0.4$, because the multiline structure is suboptimal for insuree 2. For $\rho = 0.3$, $\rho = 0.2$ and $\rho = 0.1$, the multiline structure dominates since it is possible to improve expected utility for insuree 2, as well as for insuree 1. Parameters: $p = 0.25$, $q = 0.65$, $\delta = 0.2$, $\beta = 7$.*

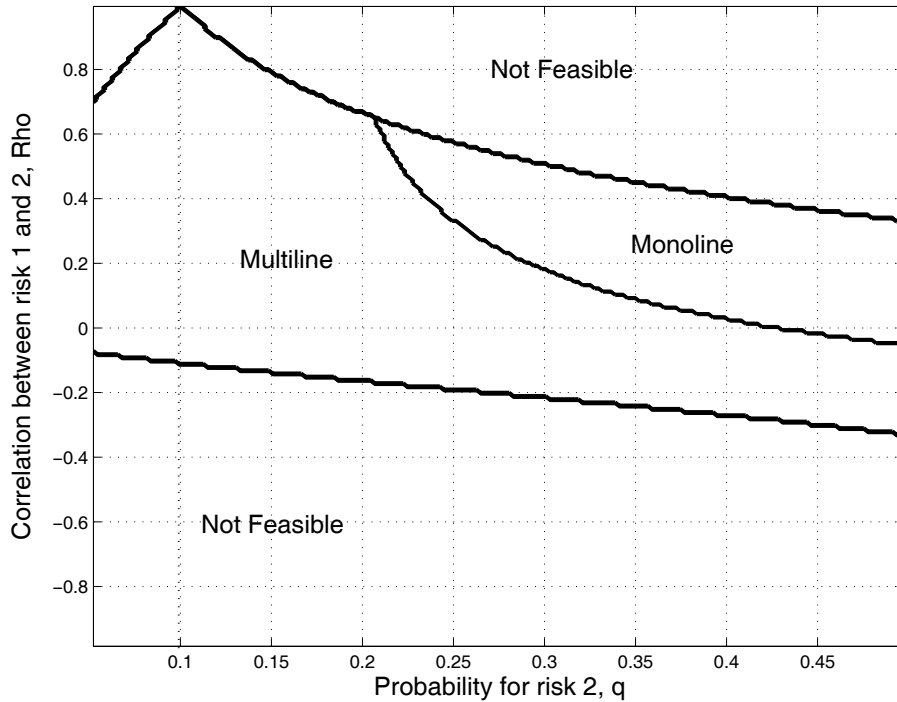


Figure 4: Regions of q and ρ , in which monoline and multiline structure is optimal. All else equal: Increasing ρ (correlation), given q makes monoline structure more likely. Increasing asymmetry of risks ($q - p$) also makes monoline structure more likely. Correlations can not be arbitrary for the two (Bernoulli) risks, so there are combinations of q and ρ that are not feasible. Parameters: $p = 0.1$, $\delta = 0.01$, $\beta = 1.2$.

occur respectively, as a function of q and ρ . We use the parameter values $p = 0.1$, $\beta = 1.2$, $\delta = 0.01$ and $t = 1$. In line with our previous discussion and summarized in Implication 1, the Figure shows that, all else equal, increasing the correlation decreases the prospects for a multiline solution. Also, increasing the asymmetry ($q - p$) between risks decreases the prospects for a multiline outcome.

Thus, in line with Implication 1, we find that multiline insurers choose lines in which

- Losses are uncorrelated/have low correlation.
- Loss distributions are similar/not too asymmetric.

4 Concluding remarks

This paper develops a model of the insurance market under the assumptions of costly capital, limited liability and perfect competition. The premium setting and capital allocations build on the no-arbitrage, option-based, technique, first introduced for a multiline insurer by Phillips, Cummins, and Allen (1998) and Myers and Read (2001), and further developed in Ibragimov, Jaffee, and Walden (2008b). These results apply for any given monoline or multiline industry structure.

The unique contribution of this paper is that it develops a framework to determine the optimal industry structure in terms of which insurance lines are provided by monoline versus multiline insurers. We employ an equilibrium concept based on a criterion of Pareto efficiency within a competitive industry. Pareto dominated structures are eliminated by new entrants that offer a preferred structure. A resulting equilibrium is robust to the entry of any new monoline provider. Capital levels and premiums are set optimally for the given equilibrium industry structure.

We derive two important properties for any such equilibrium. First, we show that the multiline structure dominates when the benefits of diversification are achieved because the underlying lines are numerous and uncorrelated. Second, we show that the monoline structure may be the efficient form when the risks are difficult to diversify because they are limited in number and heavy tailed, as is characteristic of the various catastrophe lines. The intuition for this result is that a multiline insurer with a portfolio of diversifiable risks will accept a catastrophe line only if that line contributes enough capital to leave the insurers expected default rate unchanged. With very heavy tails, the required capital contribution is very large, and it becomes in the best interest of the catastrophe line insurees to join a monoline insurer which holds less capital, albeit with a higher default risk.

Our results are also consistent with the observed structure of the insurance industry by lines. Consumer lines such as homeowners and auto insurance are dominated by

multiline insurers, while the catastrophe lines of bond and mortgage default insurance are available only on a monoline basis. Furthermore, it is a feature of our model that the default probability for a monoline bond or mortgage default insurer would likely be higher than it is for a multiline firm offering coverage only on highly diversifiable lines.

Appendix

Proof of Lemma 1: 1 and 2 follow immediately from the definition of P in (1). 3 is an immediate consequence of (1), and that $E[\tilde{Q}(A)] = o(1)$ for large A , which follows from $E[\tilde{L}]$ being finite. 4 follows from 2 and that $\eta' > 0$ for general distributions (see Ingersoll (1987)). ■

Proof of Proposition 1

We first prove the “only if”-part. Assume that for all $A > 0$, $P(A) \geq A$. Let x_- denote $\min(x, 0)$. For a given A , expected utility is $Eu(-P(A) + (A - \tilde{L})_-) \leq Eu(-P(A) + A - \tilde{L}) \leq Eu(-\tilde{L}) = Eu(-P(0) + (0 - \tilde{L})_-)$, so $0 \in A^*$.

For the “if”-part: Assume that there is an A such that $P(A) < A$. Obviously, $A > 0$, since $P(0) = 0 = A$. Now, define the “utility function” $u_q(x) = (x + q)_-$. This function is concave, but only weakly so, and not twice continuously differentiable, so it is outside the class of utility functions we are studying. However, it is easy to “regularize” u_q and get an infinitely differentiable strictly increasing and concave function that is arbitrarily close to u_q in any reasonable topology. We can do this by using the Gaussian test function, $\phi(x) = \frac{1}{2\sqrt{2\pi}}e^{-x^2/2}$ and define $\phi_\epsilon(x) = \phi(x/\epsilon)/\epsilon$. Finally, we define $u_{q,\epsilon}(x) = u_q * \phi_\epsilon = \int_{-\infty}^{\infty} u_q(y)\phi_\epsilon(x - y) dy$. Clearly, as $\epsilon \searrow 0$, $u_{q,\epsilon}$ converges to u_q . Moreover, $u_{q,\epsilon}$ is infinitely differentiable and since $u_{q,\epsilon}^{(n)} = (u_q * \phi_\epsilon)^{(n)} = u_q^{(n)} * \phi_\epsilon$, where $u_q^{(n)}$ denotes the n th derivative of u_q , it is easy to check that $u_{q,\epsilon}' > 0$ and $u_{q,\epsilon}'' < 0$ for all q and ϵ , so $u_{q,\epsilon}$ belongs to our class of utility functions.

Now, if $A > P$, then $Eu_P(-P + (A - \tilde{L})_-) = E[(A - \tilde{L})_-] > E[(P - \tilde{L})_-] = Eu_P(-\tilde{L})$, so an insuree with “utility function” u_P is strictly better off by choosing insurance. However, since $\lim_{\epsilon \searrow 0} Eu_{P,\epsilon}(-P + (A - \tilde{L})_-) = Eu_P(-P + (A - \tilde{L})_-)$ and $\lim_{\epsilon \searrow 0} Eu_{P,\epsilon}(-\tilde{L}) = Eu_P(-\tilde{L})$, for ϵ small enough, the strict inequality also holds for a $u_{P,\epsilon}$, which belongs to our class of utility functions. Thus, insurance is optimal for an insuree with such a utility function. ■

Proof of Proposition 2

1. Clearly, since $\delta > 0$, choosing $A = Z$ always dominates choosing $A > Z$, as the insurance payoffs are identical in both states of the world, but the cost of internal capital is higher if $A > Z$ than if $A = Z$. Thus, the solution, which is unique, since the objective function is strictly concave, must lie in $[0, Z]$.

Define $q = \delta + p$. The first order condition from (4) is

$$(1 - p)qu'(-qA^*) = p(1 - q)u'((1 - q)A^* - Z),$$

which, when the function $b_Z(A) \stackrel{\text{def}}{=} \frac{u'(-qA)}{u'((1 - q)A - Z)}$ is defined is equivalent to

$$b_Z(A^*) = \frac{p}{1 - p} \times \frac{1 - q}{q}, \quad (11)$$

where, since $q > p$, the right hand side is strictly less than 1. Now, since u is strictly concave and twice continuously differentiable, it follows that $b_Z(A)$ is strictly increasing in A , and since $b_Z(Z) = 1 > \frac{p}{1 - p} \times \frac{1 - q}{q}$, the maximum must indeed be realized for $A^* < Z$. The monotonicity of the utility function, u , obviously implies that b_Z is positive.

2. and 3. Since the r.h.s. of (11) does not depend on Z , and b_Z is increasing in A regardless of

Z , it is sufficient to show that $\frac{\partial b_Z(A)}{\partial Z} < 0$. Differentiating b_Z with respect to Z leads to

$$b'_Z = -ARA((1-q)A - Z)b_Z,$$

where the absolute risk aversion is defined as $ARA(x) \stackrel{\text{def}}{=} -\frac{u''(x)}{u'(x)} > 0$, so it is indeed the case that $b'_Z < 0$.

4. and 5. We have $P_Q(A) = p(Z - A)$ for $A < Z$. Therefore, $\frac{dP_Q}{dZ} = p\left(1 - \frac{dA^*}{dZ}\right)$. So, if $\frac{dA^*}{dZ} < 1$, then $\frac{dP_Q}{dZ} > 0$.

From the implicit function theorem, it follows that

$$\frac{dA^*}{dZ} = -\frac{\frac{\partial b_Z}{\partial Z}}{\frac{\partial b_Z}{\partial A}} = \frac{ARA((1-q)A - Z)b_Z}{(qARA(-qA) + (1-q)ARA((1-q)A - Z))b_Z} = \frac{ARA((1-q)A - Z)}{qARA(-qA) + (1-q)ARA((1-q)A - Z)}.$$

Now, since $(1-q)A - Z < -qA$, and for an agent with DARA preferences ARA is positive and decreasing, $\frac{dA^*}{dZ}$ is therefore less than 1, and 4. follows. A similar argument for IARA preferences leads to 5. We are done. ■

Proof of Proposition 3: Choose $A \in A_2^*$, such that $A > \mu_{L_2}$. The utility of insuring a risk \tilde{L}_2 is $Eu[-\mu_L - \delta A + (\mu_Q - \tilde{Q})] = Eu[-\mu_L - \delta A + (A - \tilde{L})_- - E[(A - \tilde{L})_-]]$. Here, we use the notation $x_- = \min(x, 0)$. Since $E[\tilde{L}_1] = E[\tilde{L}_2] = \mu_L$, a sufficient condition for risk 1 to be preferred is that $Eu((A - \tilde{L}_1)_- - E[(A - \tilde{L}_1)_-]) \geq Eu((A - \tilde{L}_2)_- - E[(A - \tilde{L}_2)_-])$, which in turn is implied by

$$(A - \tilde{L}_1)_- \succeq (A - \tilde{L}_2)_- + z, \quad (12)$$

where $z = E[(A - \tilde{L}_1)_-] - E[(A - \tilde{L}_2)_-]$, so if $g_1 \stackrel{\text{def}}{=} (A - \tilde{L}_1)_-$, and $g_2 \stackrel{\text{def}}{=} (A - \tilde{L}_2)_- + z$, so a sufficient condition is that

$$g_1 \succeq g_2. \quad (13)$$

We wish to apply the integral condition. For $t < 0$, we have that the c.d.f. of g_1 is

$$G_1(t) = F_1(t - A), \quad (14)$$

and for $t \geq 0$, $G_1(t) = 1$. The c.d.f. of g_2 , for $t < z$ is

$$G_2(t) = F_2(t - A - z), \quad (15)$$

and for $t \geq z$, $G_2(t) = 1$.

Since, if $z \geq 0$, $G_1(t) \geq G_2(t)$ when $t \geq 0$, and if $z < 0$, $G_1(t) \geq G_2(t)$ when $t > z$, and moreover, for a general r.v., \tilde{X} , with support on $[A, B]$ and c.d.f. F , $\int_A^B F(t)dt = [F(t)t]_A^B - E[\tilde{X}]$, the integral condition that ensures (13) is

$$\int_{-\infty}^t G_1(s)ds \leq \int_{-\infty}^t G_2(s)ds, \quad t < \min(0, z). \quad (16)$$

That is, it is enough to ensure that the integral condition is satisfied for $t < \min(0, z)$, instead of on the whole support of g_1 and g_2 , which is $t \leq \max(0, z)$.

However, (16) can be rewritten as

$$\int_{-\infty}^t F_1(s)ds \leq \int_{-\infty}^t F_2(s-z)ds, \quad t < -A + \min(0, z), \quad (17)$$

and since $z \leq \mu_L$, a sufficient condition for (16) is

$$\int_{-\infty}^t F_1(s)ds \leq \int_{-\infty}^t F_2(s-\mu_L)ds, \quad t < -A,$$

i.e.,

$$\int_{-\infty}^t F_1(s)ds \leq \int_{-\infty}^{t-\mu_L} F_2(s)ds, \quad t < -A,$$

and since $A \geq \mu_L$, this is implied by the even stronger condition

$$\int_{-\infty}^t F_1(s)ds \leq \int_{-\infty}^{t-\mu_L} F_2(s)ds, \quad t < -\mu_L. \quad (18)$$

Thus, the insuree prefers risk 1 over 2 at A , when (18) is satisfied, which is the optimal value of capital for \tilde{L}_2 . Now, it may well be that the insuree can be even better off for another value of capital, A' , under risk \tilde{L}_1 , e.g., $A' = 0$. However, this will just make \tilde{L}_1 even more preferred compared with \tilde{L}_2 . We are done. \blacksquare

Proof of Proposition 4: To simplify the notation, in this proof we write L_i instead of \tilde{L}_i , L instead of \tilde{L} , and μ_i instead of μ_{L_i} . Moreover, C represents an arbitrary positive, finite constant, i.e., that does not depend on N . The condition that u is twice continuously differentiable with u'' bounded by a polynomial of degree q implies that

$$|u''(z)| \leq C(1 + |z|^q), \quad z \leq 0, \quad (19)$$

for some constant, $C > 0$.

Moreover, condition $E|L_i|^p \leq C$, together with Jensen's inequality implies that $E|L_i - \mu_i|^p \leq C$, $\sigma_i^2 \leq (E|L_i - \mu_i|^p)^{2/p} \leq C$. Using the Rosenthal inequality for sums of independent mean-zero random variables, we obtain that, for some constant $C > 0$,

$$E\left|\sum_{i=1}^N (L_i - \mu_i)\right|^p \leq C \max\left(\sum_{i=1}^N E|L_i - \mu_i|^p, \left(\sum_{i=1}^N \sigma_i^2\right)^{p/2}\right), \quad (20)$$

and, thus,

$$\begin{aligned}
N^{-p} E \left| \sum_{i=1}^N (L_i - \mu_i) \right|^p &\leq CN^{-p} \max \left(\sum_{i=1}^N E |L_i - \mu_i|^p, \left(\sum_{i=1}^N \sigma_i^2 \right)^{p/2} \right) \\
&\leq CN^{-p/2} \max \left(N^{-p/2} \times CN, N^{-p/2} \times (CN)^{p/2} \right) \\
&\leq CN^{-p/2} \\
&\rightarrow 0,
\end{aligned} \tag{21}$$

as $N \rightarrow \infty$, where the second to last inequality follows since $p \geq 2$.

Take $A = \sum_{i=1}^N \mu_i$. This represents a fully multiline insurer, who chooses internal capital to be equal to total expected losses. Denote

$$y_i = -\mathcal{K}_i(\mathcal{S}) = L_i \max \left(1 - \frac{A}{\sum_{i=1}^N L_i}, 0 \right) = L_i \max \left(1 - \frac{\sum_{i=1}^N \mu_i}{\sum_{i=1}^N L_i}, 0 \right).$$

From (3) and (9), the expected utility of inseree i is then $Eu(-\mu_i(1+\delta) + Ey_i - y_i + \delta\mu_i(b_i - 1))$. Here,

$$b_i \stackrel{\text{def}}{=} E \left[\frac{L_i}{\mu_i} \times \frac{\sum_{i=1}^N \mu_i}{\sum_{i=1}^N L_i} \times \frac{\tilde{V}}{E[\tilde{V}]} \right], \tag{22}$$

and $\tilde{V} = \left(\sum_{i=1}^N L_i - \sum_{i=1}^N \mu_i L_i \right)_+$, in line with the definition of the binary default option in section 2.2. Clearly, for large N , $b_i \rightarrow 1$ uniformly over the i 's, i.e., $\lim_{N \rightarrow \infty} \max_{1 \leq i \leq N} |1 - b_i| = 0$, so $Eu(-\mu_i(1+\delta) + Ey_i - y_i + \delta\mu_i(b_i - 1)) \rightarrow Eu(-\mu_i(1+\delta) + Ey_i - y_i)$ uniformly.

Using a Taylor expansion of order one around $-\mu_i(1+\delta)$, and the polynomial bound, (19), for u'' , we get

$$\begin{aligned}
u(-\mu_i(1+\delta) + Ey_i - y_i) &= u(-\mu_i(1+\delta)) + u'(-\mu_i(1+\delta))(Ey_i - y_i) + \frac{u''(\xi(y_i))}{2}(y_i - Ey_i)^2, \\
\xi(y_i) &\in [-\mu_i(1+\delta), -\mu_i(1+\delta) + Ey_i - y_i], \forall y_i.
\end{aligned}$$

Therefore,

$$Eu(-\mu_i(1+\delta) + Ey_i - y_i) = u(-\mu_i(1+\delta)) + \frac{1}{2} E(u''(\xi(y_i))(y_i - Ey_i)^2),$$

so

$$|Eu(-\mu_i(1+\delta) + Ey_i - y_i) - u(-\mu_i(1+\delta))| \leq C \times E((1 + |y_i|^q)(y_i - Ey_i)^2) \leq C' \times (E|y_i| + E|y_i|^{2+q}).$$

If the right hand side is small, then the expected utility is close to $u(-\mu_i(1+\delta))$. To complete the proof, it thus suffices to show that

$$E|y_i|^{2+q} \rightarrow 0 \tag{23}$$

as $N \rightarrow \infty$, where the speed of convergence does not depend on i , i.e., the convergence is uniform over i .

By Jensen's inequality, evidently, $E|L_i|^p \leq C$ for $p = 2 + q + \epsilon$ with $0 < \epsilon \leq 2 + q$. For such p and ϵ , using the obvious bound $\max \left(1 - \frac{\sum_{i=1}^N \mu_i}{\sum_{i=1}^N L_i}, 0 \right) \leq 1$ and Hölder's inequality, we get, under the

conditions of the proposition,

$$\begin{aligned}
E|y_i|^{2+q} &= E\left|L_i \max\left(1 - \frac{\sum_{i=1}^N \mu_i}{\sum_{i=1}^N L_i}, 0\right)\right|^{2+q} \leq E|L_i|^{2+q} \left(\max\left(\frac{\sum_{i=1}^N (L_i - \mu_i)}{\sum_{i=1}^N L_i}, 0\right)\right)^\epsilon \leq \\
&E|L_i|^{2+q} \left|\frac{\sum_{i=1}^N (L_i - \mu_i)}{\sum_{i=1}^N L_i}\right|^\epsilon \leq (E|L_i|^p)^{\frac{2+q}{p}} \left(E\left|\frac{\sum_{i=1}^N (L_i - \mu_i)}{\sum_{i=1}^N L_i}\right|^p\right)^{\epsilon/p} \leq \\
&C \left(E\left|\frac{\sum_{i=1}^N (L_i - \mu_i)}{\sum_{i=1}^N \mu_i}\right|^p\right)^{\epsilon/p} \leq C \left(N^{-p} E\left|\sum_{i=1}^N (L_i - \mu_i)\right|^p\right)^{\epsilon/p}. \quad (24)
\end{aligned}$$

From (24) and (21) it follows that (23) indeed holds, and, since the constants do not depend on i , the convergence is uniform over i .

We now go from expected utility to certainty equivalents. Expected utility is $u(-\mu_i(1+\delta)) - \epsilon = u(-\mu_i(1+\delta) - c) = u(-\mu_i(1+\delta)) - cu'(\xi)$, where $\xi \in (-\mu_i(1+\delta) - c, -\mu_i(1+\delta))$, so $c = \epsilon/u'(\xi) \leq \epsilon/u'(-\mu_i(1+\delta)) \leq \epsilon/u'(0)$, and $\epsilon \rightarrow 0$ therefore implies that $c \rightarrow 0$. The proof is complete. \blacksquare

Proof of Proposition 5:

By Taylor expansion, for all x, y , $u(x+y) = u(x) + u'(x)y + u''(\zeta)\frac{y^2}{2}$, where ζ is a number between x and $x+y$. Since u'' is bounded away from zero: $-u'' \geq C > 0$, we, therefore, get

$$u(x+y) \leq u(x) + u'(x)y - C\frac{y^2}{2} \quad (25)$$

for all x, y . Using inequality (25), in the notations of the proof of Proposition 4, we obtain

$$\begin{aligned}
&Eu(-\mu_i(1+\delta) + Ey_i - y_i) \leq \\
&E\left[u(-\mu_i(1+\delta)) + u'(-\mu_i(1+\delta))(Ey_i - y_i) - C\frac{(y_i - Ey_i)^2}{2}\right] = \\
&u(-\mu_i(1+\delta)) - CVar(y_i). \quad (26)
\end{aligned}$$

Consequently, if $u(-\mu_i(1+\delta)) - Eu(-\mu_i(1+\delta) + Ey_i - y_i) < \epsilon$, then $Var(y_i) < \epsilon' = \epsilon/C$. We therefore wish to show that, for a fixed number of risks, N , $Var(y_i) \geq \epsilon_N > 0$.

We need the following two lemmas:

Lemma 2 For $x_2 \geq x_1$, if X is a random variable, such that $\mathbb{P}(X \leq x_1) = a$ and $\mathbb{P}(X \geq x_2) = b$, then $Var(X) \geq \frac{\min(a,b)(x_2-x_1)^2}{4}$.

Proof: Define $e = \frac{x_2-x_1}{2} \geq 0$. Define $M = E(X)$ and assume that $M \geq x_1 + e$. Moreover, let ϕ denote X 's p.d.f. Then $Var(X) = \int (x-M)^2 \phi(x) dx \geq \int_{x \leq x_1} (x-M)^2 \phi(x) dx \geq \int_{x \leq x_1} (x-x_1-e)^2 \phi(x) dx \geq e^2 a = \frac{a(x_2-x_1)^2}{4}$. A similar argument shows that if $M < x_1 + e$, then $Var(X) \geq \frac{b(x_2-x_1)^2}{4}$, and since either $M \geq x_1 + e$ or $M < x_1 + e$ the lemma follows.

Lemma 3 If X is a random variable, such that $0 \leq X \leq C < \infty$, $E(X) = \mu$ and $Var(X) = \sigma^2 > 0$, then there are constants $d > 0$ and $\epsilon > 0$, that only depend on C , μ and σ^2 , such that $\mathbb{P}(X \leq \mu - \epsilon) \geq d$ and $\mathbb{P}(X \geq \mu + \epsilon) \geq d$.

Proof Let ϕ be X 's p.d.f. For a small $\epsilon > 0$, we have

$$\begin{aligned}
\sigma^2 &= \int_0^C (x - \mu)^2 \phi(x) dx \\
&= \int_{|x-\mu|<\epsilon} (x - \mu)^2 \phi(x) dx + \int_{|x-\mu|\geq\epsilon} (x - \mu)^2 \phi(x) dx \\
&\leq \epsilon^2 + \mu \int_{x\leq\mu-\epsilon} |x - \mu| \phi(x) dx + (C - \mu) \int_{x\geq\mu+\epsilon} |x - \mu| \phi(x) dx.
\end{aligned} \tag{27}$$

Moreover, $\int_{|x-\mu|<\epsilon} (x - \mu) \phi(x) dx + \int_{|x-\mu|\geq\epsilon} (x - \mu) \phi(x) dx = 0$, and $\left| \int_{|x-\mu|<\epsilon} (x - \mu) \phi(x) dx \right| \leq \epsilon$, so

$$\int_{x\geq\mu+\epsilon} |x - \mu| \phi(x) dx + \epsilon \geq \int_{x\leq\mu-\epsilon} |x - \mu| \phi(x) dx.$$

Plugging this into (27) yields

$$C \int_{x\geq\mu+\epsilon} |x - \mu| \phi(x) dx \geq \sigma^2 - \epsilon^2 - \mu\epsilon.$$

However, since $\int_{x\geq\mu+\epsilon} |x - \mu| \phi(x) dx \leq (C - \mu) \int_{x\geq\mu+\epsilon} \phi(x) dx = (C - \mu) \mathbb{P}(X \geq \mu + \epsilon)$, we arrive at

$$\mathbb{P}(X \geq \mu + \epsilon) \geq \frac{\sigma^2 - \epsilon^2 - \epsilon\mu}{C(C - \mu)}.$$

A similar argument implies that

$$\int_{x\leq\mu-\epsilon} |x - \mu| \phi(x) dx + \epsilon \geq \int_{x\geq\mu+\epsilon} |x - \mu| \phi(x) dx,$$

which, when plugged into (27) yields

$$C \int_{x\leq\mu-\epsilon} |x - \mu| \phi(x) dx \geq \sigma^2 - \epsilon^2 - \epsilon(C - \mu),$$

and since $\int_{x\leq\mu-\epsilon} |x - \mu| \phi(x) dx \leq \mu \int_{x\geq\mu+\epsilon} \phi(x) dx = \mu \mathbb{P}(X \leq \mu - \epsilon)$, we arrive at

$$\mathbb{P}(X \leq \mu - \epsilon) \geq \frac{\sigma^2 - \epsilon^2 - \epsilon(C - \mu)}{C\mu}.$$

Thus, by defining

$$d \stackrel{\text{def}}{=} \min \left(\frac{\sigma^2 - \epsilon^2 - \epsilon\mu}{C(C - \mu)}, \frac{\sigma^2 - \epsilon^2 - \epsilon(C - \mu)}{C\mu} \right), \tag{28}$$

the lemma follows.

We note that the condition in proposition 4, that $EL_i \geq C'$ for some $C' > 0$, actually is implied by the conditions that $Var(L_i) > C_1$ and that $L_i \leq C_0$, by the following lemma

Lemma 4 *If X is a random variable, such that $0 \leq X \leq C_0 < \infty$, then $E(X) \geq \frac{\text{Var}(X)}{C_0}$.*

Proof: Define $e = E(X)$ and let ϕ denote X 's p.d.f. We have $\text{Var}(X) = \int_0^C x^2 \phi(x) dx - e^2 \leq \int_0^C x^2 \phi(x) dx \leq C \int_0^C x \phi(x) dx = C \times E[X]$.

Now, from the conditions of proposition 5, it follows that the L_i 's satisfy the conditions of lemma 3, and that ϵ and d can be chosen not to depend on i . Moreover, $y_i = L_i \max\left(1 - \frac{\sum_i \mu_i}{\sum_i L_i}, 0\right)$, so $\mathbb{P}(y_i \leq 0) \geq \mathbb{P}(\sum_i L_i \leq \sum_i \mu_i) \geq \prod_{i=1}^N \mathbb{P}(L_i \geq \mu_i) \geq \prod_{i=1}^N \mathbb{P}(L_i \geq \mu_i + \epsilon) \geq d^N$, where d is defined in (28).

Similarly, if $L_i \geq \mu_i + \epsilon$ for all i , then $y_i = L_i \left(1 - \frac{\sum_i \mu_i}{\sum_i L_i}\right) \geq (\mu_i + \epsilon) \left(1 - \frac{\sum_i \mu_i}{\sum_i \mu_i + N\epsilon}\right) = (\mu_i + \epsilon) \times \left(1 - \frac{1}{1 + \frac{N\epsilon}{\sum_i \mu_i}}\right) \geq \mu_i \times \left(1 - \left(1 - \frac{N\epsilon}{N\epsilon}\right)\right) = \mu_i \times \frac{\epsilon}{C}$, so $\mathbb{P}(y_i \geq \frac{\mu_i \epsilon}{C}) \geq \prod_{i=1}^N \mathbb{P}(L_i \geq \mu_i + \epsilon) \geq d^N$.

Now, from lemma 4, $\mu_i \geq \frac{\sigma^2}{C}$ for all i , so we have $\mathbb{P}(y_i \geq \frac{\sigma^2 \epsilon}{C^2}) \geq d^N$. Therefore, y_i satisfies all the conditions of lemma 2, with $x_1 = 0$, $x_2 = \frac{\sigma^2 \epsilon}{C^2}$, $a = b = d^N$, and therefore

$$\text{Var}(y_i) \geq C_N, \quad \text{where } C_N = \frac{\sigma^4 \epsilon^2}{4C^4} \left(\min \left(\frac{\sigma^2 - \epsilon^2 - \epsilon\mu}{C(C - \mu)}, \frac{\sigma^2 - \epsilon^2 - \epsilon(C - \mu)}{C\mu} \right) \right)^N > 0.$$

The constant, C_N , depends on N , σ^2 and C , but not on the specific distributions of the L_i 's, so for a fixed N ,

$$Eu(-\mu_i(1 + \delta) + Ey_i - y_i) \leq u(-\mu_i(1 + \delta)) - C \times C_N. \quad (29)$$

The argument, so far, has been for $Eu(\mu_i(1 + \delta) + Ey_i - y_i)$, whereas the utility is $Eu(\mu_i(1 + \delta) + Ey_i - y_i + \delta\mu_i(b_i - 1))$, where b_i is defined in (22). However, since by definition $\sum_{i=1}^N b_i = 1$ for all N , it is clear that for all N , there must be an i , such that $b_i - 1 \leq 0$. For such an i , $Eu(\mu_i(1 + \delta) + Ey_i - y_i + \delta\mu_i(b_i - 1)) < Eu(\mu_i(1 + \delta) + Ey_i - y_i)$, so $Eu(\mu_i(1 + \delta)) - Eu(\mu_i(1 + \delta) + Ey_i - y_i) > \epsilon \Rightarrow Eu(\mu_i(1 + \delta)) - Eu(\mu_i(1 + \delta) + Ey_i - y_i + \delta\mu_i(b_i - 1)) > \epsilon$. Thus, the argument also goes through for the utility of agent.

A similar argument as in the proof of proposition 4, takes us from utilities to certainty equivalents: $u(-\mu_i(1 + \delta) - c) = u(-\mu_i(1 + \delta)) - \epsilon \Rightarrow c = \epsilon/u'(-\xi)$, $\xi \leq C_0$, so $c \geq \frac{\epsilon}{u'(-C_0)}$, so if ϵ is bounded away from 0, so is c . We are done. ■

Proof of Proposition 6: We use the same notation as in the proof of proposition 4. We prove the third result first, since it will help us in proving the other results.

3. For the fully multiline industry structure, with capital $A = \beta \sum_i \mu_i$, the utility of agent i is $Eu(-\mu_i(1 + \delta\beta) + Ey_i - y_i)$, where

$$y_i = -\mathcal{K}_i(\mathcal{S}) = L_i \max\left(1 - \frac{A}{\sum_{i=1}^N L_i}, 0\right) = L_i \max\left(1 - \beta \frac{\sum_{i=1}^N \mu_i}{\sum_{i=1}^N L_i}, 0\right). \quad (30)$$

Here, since we will use the results for non i.i.d. risks when proving (1.) and (2.), we use the i -subscript, even though it is not needed in the i.i.d. case, in which $\mu_i \equiv \mu$.

Assume that $(\mathcal{X}^{MONO}, (A_1, A_2, A_3, \dots))$ is a constrained Pareto efficient industry structure. Then, $\mathcal{S}^{MONO} = (\mathcal{X}^{MONO}, (A_1, A_1, A_1, \dots))$ is obviously Pareto equivalent, i.e., it provides the same expected utility for all insurees, i , as $(\mathcal{X}^{MONO}, (A_1, A_2, A_3, \dots))$ does. Such an industry structure is characterized by $\beta \stackrel{\text{def}}{=} A/\mu$. If we show that, regardless of β , for large enough N , and

some β^{MULTI} , the industry partition, \mathcal{X}^{MULTI} , with capital, A , where $A = \beta^{MULTI} \times \sum_i \mu_i$, Pareto dominates S^{MONO} , then we are done.

For $\beta^{MULTI} = 1$, the argument of proposition 4 implies that $Eu(-\mu_i(1 + \delta\beta^{MULTI}) + Ey_i - y_i) \rightarrow u(-\mu_i(1 + \delta))$, i.e., the expected utility converges to the ideal risk-free outcome with costly capital, as N grows, and that the convergence is uniform in i . The utility from insuring with a monoline insurer with $\beta^{MONO} = 1$, on the other hand, is $Eu(-\mu_i(1 + \delta) + Ey_i - y_i)$, where (30) implies that $y_i = \max(L_i - \beta^{MONO}\mu_i, 0)$. Now, $Ey_i - y_i$ is obviously second order stochastically dominated (SOSD) by 0, so for $\beta^{MONO} = 1$, the fully multiline partition with $\beta^{MULTI} = 1$, will obviously dominate the monoline offering. Similarly, the multiline partition with $\beta^{MULTI} = 1$ dominates any monoline offering with $\beta^{MONO} > 1$, since there is still residual risk for such a monoline offering and the total cost of internal capital is higher — effects that both make the insuree worse off.

It is also clear that for large N , β^{MULTI} for any constrained Pareto efficient outcome must lie in $[0, 1 + o(1)]$, since internal capital is costly, which imposes a linear cost of increasing β , and, by the law of large numbers, all risk eventually vanishes for $\beta^{MULTI} = 1$. Thus, there is always a constrained Pareto efficient solution in $[0, 1 + o(1)]$, given the fully multiline industry partition.

If we show that, for a given $0 < \beta < 1$, the fully multiline outcome will dominate monoline offerings with the same β , then we are done, since, regardless of a candidate β^{MONO} , choosing $\beta^{MULTI} = \beta^{MONO}$ will lead to a Pareto improvement ($\beta^{MONO} = 0$ is strictly dominated by some $\beta^{MONO} > \epsilon/\mu$ from condition 3, so we do not need to consider $\beta^{MONO} = 0$).

For the fully multiline outcome, it is clear from (30), using an identical argument as in the proof of proposition 4 that y_i converges uniformly in i to $(1 - \beta)L_i$, and the expected utility of agent i converges uniformly to

$$Eu(-\mu_i(1 + \delta\beta) + (1 - \beta)(\mu_i - L_i)). \quad (31)$$

On the other hand, for the monoline offering, the expected utility is

$$Eu(-\mu_i(1 + \delta\beta) + Ey_i - y_i), \quad (32)$$

where

$$y_i = \max(L - \beta\mu_i, 0) = (1 - \beta) \max\left(\mu_i - \frac{\mu_i - L}{1 - \beta}, 0\right). \quad (33)$$

We define $z_i \stackrel{\text{def}}{=} \mu_i - L_i \in (-\infty, \mu_i)$ and $\alpha \stackrel{\text{def}}{=} \frac{1}{1 - \beta} \in (1, \infty)$. Equations (31-33) imply that if

$$(1 - \beta)z_i \succeq (1 - \beta) (E[\max(\mu_i - \alpha z_i, 0)] - \max(\mu_i - \alpha z_i, 0)) \quad (34)$$

for all $\alpha \in (1, \infty)$, where \succeq denotes second order stochastic dominance, then the third part of proposition 6 is proved. However, (34) is equivalent to

$$z_i \succeq E[\max(\mu_i - \alpha z_i, 0)] - \max(\mu_i - \alpha z_i, 0),$$

and by defining $x_i \stackrel{\text{def}}{=} \max(\mu_i - \alpha z_i, 0)$, and $w_i \stackrel{\text{def}}{=} Ex_i - x_i$, to

$$z_i \succeq w_i.$$

We define $q_i(\alpha) \stackrel{\text{def}}{=} Ex_i$. From the definitions in section 2: $\tilde{Q}(A) = \max(L - A, 0)$ and $P_Q(A) = E[\tilde{Q}(A)]$, it follows that $q_i(\alpha) = \alpha P_Q\left(\left(1 - \frac{1}{\alpha}\right)\mu_i\right)$. Therefore,

$$w_i = q_i(\alpha) - \max(\mu_i - \alpha z_i, 0). \quad (35)$$

To show that z_i second order stochastically dominates w_i , we use the following lemma, which

follows immediately from the theory in Rothschild and Stiglitz (1970):

Lemma 5 *if z and w are random variables with absolutely continuous distributions, and distribution functions $F_z(\cdot)$ and $F_w(\cdot)$ respectively, $z \in (\underline{z}, \bar{z})$, $-\infty \leq \underline{z} < \bar{z} \leq \infty$ (a.s.) and the following conditions are satisfied:*

1. $Ez = Ew$,
2. $F_z(x) < F_w(x)$ for all $\underline{z} < x < x_0$, for some $x_0 > \underline{z}$,
3. $F_z(x) = F_w(x)$ at exactly one point, $x^* \in (\underline{z}, \bar{z})$.

Then $z \succ w$, i.e., z strictly SOSD w .

Clearly, $Ez_i = Ew_i = 0$, so the first condition of lemma 5 is satisfied. For the second condition, it follows from (35), that for $x < q_i(\alpha)$,

$$\begin{aligned} F_{w_i}(x) &= \mathbb{P}(w_i \leq x) = \mathbb{P}(q_i(\alpha) - \mu_i + \alpha z_i \leq x) \\ &= \mathbb{P}(z_i \leq (x + \mu_i - q_i(\alpha))/\alpha) \\ &= F_{z_i}((x + \mu_i - q_i(\alpha))/\alpha). \end{aligned} \quad (36)$$

Since $\alpha > 1$, it is clear that for small enough x , $(x + \mu_i - q_i(\alpha))/\alpha > x$, and therefore $F_{z_i}(x) < F_{w_i}(x)$, so the second condition is satisfied.

To show that the third condition is satisfied, we need the following lemma

Lemma 6 $q_i(\alpha) \geq \mu_i$ for all $\alpha > 1$.

Proof: Denote by ϕ , the probability distribution function (or measure) of L_i . Thus, $\mu_i = \int_0^\infty L\phi(L)dL$. We have, for $r > 0$:

$$\int_0^r L\phi(L)dL \leq r \int_0^r \phi(L)dL,$$

which leads to

$$\begin{aligned} \mu_i - \int_0^r L\phi(L)dL \geq \mu_i - r \int_0^r \phi(L)dL &\Leftrightarrow \int_r^\infty L\phi(L)dL \geq \mu_i - r \int_0^r \phi(L)dL \\ &\Leftrightarrow \int_r^\infty L\phi(L)dL + \int_0^r \phi(L)dL - r \geq \mu_i - r \\ &\Leftrightarrow \int_r^\infty L\phi(L)dL - r \left(1 - \int_0^r \phi(L)dL\right) \geq \mu_i - r \\ &\Leftrightarrow \int_r^\infty L\phi(L)dL - r \int_r^\infty \phi(L)dL \geq \mu_i - r \\ &\Leftrightarrow \int_r^\infty (L - r)\phi(L)dL \geq \mu_i - r \\ &\Leftrightarrow \frac{\mu_i}{\mu_i - r} \int_r^\infty (L - r)\phi(L)dL \geq \mu_i \\ &\Leftrightarrow \frac{\mu_i}{\mu_i - r} \int_0^\infty \max(L - r, 0)\phi(L)dL \geq \mu_i \\ &\Leftrightarrow \frac{\mu_i}{\mu_i - r} E[\max(L - r, 0)] \geq \mu_i. \\ &\Leftrightarrow \frac{\mu_i}{\mu_i - r} P_Q(r) \geq \mu_i. \end{aligned} \quad (37)$$

Now, for $r < \mu_i$, define $\alpha = \frac{\mu_i}{\mu_i - r} \in (1, \infty)$, implying that $r = (1 - \frac{1}{\alpha})\mu_i$. Then, the last line of (37) can be rewritten

$$q_i(\alpha) = \alpha P_Q \left(\left(1 - \frac{1}{\alpha}\right) \mu_i \right) \geq \mu_i.$$

This completes the proof of lemma 6.

Since $z \in (\underline{z}, \bar{z}) = (-\infty, \mu_i)$, and the p.d.f. is strictly positive, $F_{z_i}(\mu_i) = 1$ and, $F_{z_i}(x) < 1$ for $x < \mu_i$. We also note that for $x < q_i(\alpha)$, $F_{w_i}(x) < 1$ is given by (36), and for $x \geq q_i(\alpha)$, $F_{w_i}(x) = 1$. Since $\bar{z} \leq q_i(\alpha)$ (by lemma 6) the only points at which $F_{z_i}(x) = F_{w_i}(x)$ are therefore points at which $F_{z_i}(x) = F_{z_i}((x + \mu_i - q_i(\alpha))/\alpha)$, i.e., for points at which

$$x = \frac{x + \mu_i - q_i(\alpha)}{\alpha} \Rightarrow x = \frac{\mu_i - q_i(\alpha)}{\alpha - 1}. \quad (38)$$

Now, since there is a unique solution to (38), the third condition is indeed satisfied. Thus, z_i SOSD w_i , so a monoline offering is inferior for any insuree with strictly concave utility function, and therefore the third part (3.) of proposition 6 holds.

4. Since, from proposition 4, with $\beta = 1$ (i.e. with $A = \sum_i \mu_i$), the fully multiline outcome converges to the ideal risk-free outcome with costly internal capital, and this outcome strictly dominates any offering that can be provided by a monoline insurer, the fourth part (4.) of proposition 6 follows immediately.

1. It was shown in the proof of the third part of the proposition (3.) above that when the risks are i.i.d., a massively multiline industry partition Pareto dominates the monoline one. There are two complications when extending the proof to nonidentical distributions. The first, main, complication is that insurees may no longer agree on the optimal level of capital, i.e., they have different β_i 's. Some insurees may face "small" risks and thereby opt for limited capital, to save on costs of internal capital (a low β), whereas others may wish to have a close to full insurance (a high β). A fully multiline industry structure has only one β , and it may therefore be optimal to have several insurance companies, each with a different β . As the number of lines tends to infinity, however, within each such company the number of lines grows, making the industry structure massively — but not fully — multiline.

The second complication is technical: Since the risks are no longer i.i.d., and each insuree may therefore have a different β , most insurees will not be at their optimal β when insuring with a multiline firm, although they will be close. There is therefore a trade off between the utility loss of being at a suboptimal β , versus the utility gains of diversification, in the multiline partition. We need quantitative bounds, as opposed to the qualitative SOSD bounds in the third part of the proposition above, that show that when N increases, for most insurees the second effect dominates the first. This will be ensured by condition 2.

If condition 2 fails, even though each insuree may eventually wish to be insured by a multiline industry, for any fixed N , it could still be that most insurees prefer a monoline solution, e.g., if insurees $i = 1, \dots, N/2$ prefer a massively multiline insurer, whereas insurees $i = N/2 + 1, \dots, N$ prefer to insure with a fully multiline insurance company. In this case, $M(S) = N/2 + 1$, so the average number of lines converges to 2 and the industry structure is thus not massively multiline.

The expected utility of insuree $i \in X_j$, is $Eu(-\mu_i(1 + \delta\beta) + Ey_i - y_i + \beta\delta\mu_i(b_i - 1))$, where

$$y_i = -\mathcal{K}_i(S) = L_i \max \left(1 - \frac{A}{\sum_{i \in X_j} L_i}, 0 \right) = L_i \max \left(1 - \beta \frac{\sum_{i \in X_j} \mu_i}{\sum_{i \in X_j} L_i}, 0 \right), \quad (39)$$

and

$$b_i \stackrel{\text{def}}{=} E \left[\frac{L_i}{\mu_i} \times \frac{\sum_{i \in X_j} \mu_i}{\sum_{i \in X_j} L_i} \times \frac{\tilde{V}}{E[\tilde{V}]} \right]. \quad (40)$$

A similar argument as in proposition 4 implies that $b_i \rightarrow 1$ uniformly, so for large $|X_j|$, we can study $Eu(-\mu_i(1 + \delta\beta) + Ey_i - y_i) \rightarrow Eu(-\mu_i(1 + \delta\beta) + (1 - \beta)(\mu_i - L_i))$.

The following two lemmas are needed to show the result.

Lemma 7 *Under the conditions of proposition 6.1, there is a $C > 0$, that does not depend on i , such that for all $0 \leq \beta \leq 1$,*

$$\left| \frac{\partial Eu(-\mu_i(1 + \delta\beta) + (1 - \beta)(\mu_i - L_i))}{\partial \beta} \right| \leq C. \quad (41)$$

Proof Define $Z(\beta) \stackrel{\text{def}}{=} Eu(-\mu_i(1 + \delta\beta) + (1 - \beta)(\mu_i - L_i))$. It immediately follows that Z is concave, so $\max_{0 \leq \beta \leq 1} |Z'(\beta)|$ is realized at either $\beta = 0$ or $\beta = 1$. The derivative of Z is

$$Z'(\beta) = E[(-(1 + \delta) + L_i)u'(-\mu_i(1 + \delta\beta) + (1 - \beta)(\mu_i - L_i))],$$

so $|Z'(0)| = |E[(-1 - \delta) + L_i]u'(-L_i)| \leq C(|E[u'(0) + L_i u''(-L_i)] + |E[L_i u'(0) + L_i^2 u''(-L_i)]| \leq C(|u'(0)(\mu_i + 1) + E[(L_i + L_i^2)u''(-L_i)]|) \leq C$. Here, the last inequality follows from the assumptions in proposition 4: Since $|u''(-x)| \leq c_1 + c_2 x^q$ and $|E[L_i^2 u''(-L_i)]| \leq E[L_i^2(c_1 + c_2 L_i^q)] \leq c_1 \times E[L_i^2] + c_2 \times E[L_i^{2+q}] \leq C'$.

Thus, $|Z'(0)|$ is bounded by a constant, and moreover, since a Taylor expansion yields that $|Z'(1)| \leq |Z'(0)| + \max_{0 \leq \beta \leq 1} |Z''(\beta)|$, as long as $|Z''(\beta)|$ is bounded in $0 \leq \beta \leq 1$ (independently of i), $|Z'(1)|$ will also be bounded by a constant. We have

$$\begin{aligned} |Z''(\beta)| &= |E[(-(1 + \delta) + L_i)^2 u''(-\mu_i(1 + \delta\beta) + (1 - \beta)(\mu_i - L_i))]| \\ &\leq |E[(-(1 + \delta) + L_i)^2 u''(-\mu_i(1 + \delta) - L_i)]| \\ &\leq E[(c_1 + c_2 L_i + c_3 L_i^2)(c_4 + c_5 L_i^q)] \leq C. \end{aligned}$$

The lemma is proved.

Lemma 7 immediately implies that for β close to the optimal β^* , the utility will be close to the utility at the optimum, since $|Z(\beta + \epsilon) - Z(\beta)| \leq C\epsilon$. The second condition, that the utility provided from the asymptotic fully multiline company is bounded away from the utility provided by the monoline insurer, for a fixed β , follows from the following lemma.

Lemma 8 *If condition 2 is satisfied, then for any $\epsilon > 0$, there is a constant, $C > 0$, such that, for all $\beta \in [\epsilon, 1)$,*

$$Eu(-\mu_i(1 + \delta\beta) + (1 - \beta)(\mu_i - L_i)) - Eu(-\mu_i(1 + \delta\beta) + Ey_i - y_i) \geq C. \quad (42)$$

Here, $y_i = \max(L_i - \beta\mu_i, 0)$, is the payoff of the option to default for the monoline insurer, with capital $\beta\mu_i$.

Proof: In the notation of the proof of the third result (3.), equation (42) can be rewritten as $Eu(-c + z_i(1 - \beta)) \geq Eu(-c + w_i(1 - \beta)) + C$, where $c \stackrel{\text{def}}{=} -\mu_i(1 + \delta\beta)$, $z_i \stackrel{\text{def}}{=} (\mu_i - L_i)$ and $w_i \stackrel{\text{def}}{=} \frac{1}{1-\beta}(Ey_i - y_i)$. We know from the proof that z_i SOSD w_i , so $Eu(-c + z_i(1 - \beta)) > Eu(-c + w_i(1 - \beta))$, but we need a bound that is independent of i .

From (36), we know that

$$F_{w_i} - F_{z_i} = F_{z_i}((x + \mu_i - q_i(\alpha))/\alpha) - F_{z_i}(x) \quad (43)$$

Here, $\alpha = \frac{1}{1-\beta} \in [\frac{1}{\epsilon}, \infty)$. We know from the definition of q_i that $q_i(\alpha) \leq \alpha \times \mu_i$. The conditions of proposition 4, immediately imply that $\mu_i \leq C$ for some C independent of i , so $q_i(\alpha)$ is bounded for each α , independently of i , by $\alpha \times C_0$. If we denote by F_i , the c.d.f. of $-L_i$, then, since $z_i = \mu_i - L_i$, it follows that $F_{z_i}(x) = F_i(x - \mu_i)$. The risk faced by the insuree is $\frac{1}{\alpha}z_i$ in the asymptotic multiline case, and $\frac{1}{\alpha}w_i$ in the monoline case. Therefore, (43) implies that

$$e(x, \alpha) \stackrel{\text{def}}{=} F_i(x + \mu_i - q_i(\alpha)) - F_i(\alpha x)$$

is the difference between the c.d.f.'s of the monoline and fully insured offerings for largely negative x . Since $q_i(\alpha) \leq C_0\alpha$, $e(x, \alpha) \geq F_i(x - C_0\alpha) - F_i(\alpha x)$. we can choose x negative enough so that $h(-x) < \frac{\alpha x - 1}{x - C_0\alpha}$. Then, for $x' < x$, from condition 2, it follows that

$$e(x', \alpha) \geq g(-h(-x')(x' - C_0\alpha)) - g(-\alpha x') \geq g(-\alpha x' - 1) - g(-\alpha x').$$

This bound is independent of i , and since g is continuous and strictly decreasing, for each α it is the case that on $[-x - 1, -x]$,

$$F_i(x + \mu_i - q_i(\alpha)) - F_i(\alpha x) > \epsilon_\alpha > 0. \quad (44)$$

We have, for a general risk, R , with c.d.f. F_R , and support on $(-\infty, r)$, where R is thin-tailed enough so expected utility is defined,

$$Eu[R] = \int_{-\infty}^r u(x)dF_R = u(r) - \int_{-\infty}^r u'(x)F_R(x)dx. \quad (45)$$

Now, since u is strictly concave, u' is decreasing, but positive, so if, $E[R_1] = E[R_2]$, $F_{R_1} > F_{R_2} + \epsilon$, on $[a, b]$, $\epsilon > 0$, and if F_{R_1} and F_{R_2} only cross at one point, then $Eu[R_1] - Eu[R_2] = \int_{-\infty}^r u'(x)(F_{R_2}(x) - F_{R_1}(x))dx \geq (\inf_{x \leq r} |u''(x)|) \times \epsilon \times \frac{b-a}{2}$.

Therefore, (44), implies that $Eu(-c + z_i(1 - \beta)) \geq Eu(-c + w_i(1 - \beta)) + C_\alpha$, and since ϵ_α is continuous in α , we can take the minimum over $\alpha \in [1, 1/\epsilon]$, to get a bound that does not depend on α . The lemma is proved.

We have thus shown that, away from $\beta = 0$, the asymptotic fully multiline solution is uniformly better (in i) than the monoline solution, for each β (lemma 8), and that, as long as a β close to the optimal β is offered, the decrease in utility for the insuree is small (lemma 7). This, together with the uniform convergence toward the asymptotic risk as the number of lines covered by an insurer grows, implies that there is an $\epsilon^* > 0$ and a $C^* < \infty$, such that any firm with a β within ϵ^* distance from the optimal β for the monoline offering, and with at least C^* insurees, will make all insurees strictly better off than the monoline insurer.

Now, since $\beta \in [\epsilon, 1]$ is in a compact interval, as the number of lines grows, most insurees will be close to many insurees, making massively multiline offerings feasible for the bulk of the insurees. In fact, defining $T_N(i, \epsilon)$ to be the number of insurees, who have β 's within ϵ distance from insuree i , i.e., for which $|\beta_i - \beta_j| \leq \epsilon$, in the economy with insurees $1 \leq i \leq N$, it is the case that $\forall \epsilon > 0$, and $\forall C < \infty$, all but a bounded number of agents belong to $\{i : T_N(i, \epsilon) \geq C\}$, due to a standard compactness argument. If this were not the case, there would be a sequence of insurees, $i_j, j = 1, \dots$, with β 's not close to other insurees, and since such a sequence must have a limit point, there must be an agent with more than C neighbors within ϵ distance, contradicting the original assumption.

Now, it is clear that for agents in $\{i : T_N(i, \epsilon) \geq C\}$, as N grows, multiline solutions will be optimal. Specifically, for $\epsilon = \epsilon^*$, a sequence of $C \rightarrow \infty$ can be chosen, and an N large enough, such that $\{i : T_N(i, \epsilon) \geq C\}/N > 1 - \delta$, for arbitrary $\delta > 0$. Then it follows immediately that all insurees can be covered by insurers with at least $C/2$ insurees, where insurer n chooses $\beta_n = n\epsilon^*/2$. This leads to a massively multiline structure, since the average number of insurees is greater than or equal to $(1 - \delta)C/2$, and, as we have shown, it Pareto dominates the monoline industry partition. We are done.

2. The proposition follows immediately from the compactness argument made toward the end of the proof of the first result, (1.). If there would be a sequence of insurees, i_j , $j = 1, \dots$, for which it is Pareto optimal to insure with a monoline insurer, there must be a limit point, i.e., a $\beta^* \in [\epsilon, 1]$ for which, for each $\epsilon > 0$, there is an infinite number of insurees — each insuree having optimal capital β_{i_j} — such that $|\beta_{i_j} - \beta^*| < \epsilon$. However, the argument in (1.) then immediately implies that a multiline firm with $\beta^{MULTI} = \beta^*$ can make a Pareto improvement, contradicting the original assumption that the industry partition was Pareto efficient. ■

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